MATH 3E03, Section 01
Instructor: Deirdre Haskell

ABSTRACT ALGEBRA
MCMASTER UNIVERSITY FINAL EXAM: 11 December 2004

THIS EXAMINATION PAPER CONTAINS TWO PAGES AND FIVE QUESTIONS, NUMBERED I THROUGH V. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE EXAMINATION IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

Duration of exam: three hours.

Any calculator is allowed.

Answer all the questions below.

I. (3 points each) Give careful definitions for ALL of the following five terms in italics.

1) The set G with a binary operation on it is a *group*.
2) The function $\alpha : A \to A$ is a *permutation*.
3) φ is an *isomorphism* from the group G to the group G'.
4) The *center* of a group G.
5) A *Sylow p-subgroup* of the group G.

II. (3 points each) Give examples for ALL of the five following. You do not need to prove that your example has the stated properties, but be sure to state your example clearly and completely.

1) a non-trivial homomorphism on a group which is not an isomorphism
2) a non-abelian group and a proper, non-trivial normal subgroup
3) a function from an infinite set to itself which is 1-1 and onto, but not the identity
4) a automorphism of a group (not the identity)
5) a group with exactly one proper, nontrivial subgroup

III. (3 points each) Do ALL of the following five problems.

1) Let G be a group, H a subgroup of G and $a \in G$. Prove that $aH = H$ if and only if $a \in H$.

2) Let G be a group, $C(a)$ the centraliser of an element a in G, and $Z(G)$ the center of G. Prove the class equation: $|G| = |Z(G)| + \Sigma[G : C(a)]$, where the sum runs over representatives of the distinct, non-central conjugacy classes.

3) Let $\varphi : G \to G'$ be an isomorphism. Suppose that a generates G. Prove that $\varphi(a)$ generates G'.

4) Let $\varphi : G \to G'$ be a homomorphism. Prove that $\mathrm{Ker}(\varphi)$ is a subgroup of G and that it is normal.

5) Let n be a fixed positive integer. If $a = a' \bmod n$ and $b = b' \bmod n$, prove that $a + b = a' + b' \bmod n$ and $ab = a'b' \bmod n$. (This is a property we assumed, but never explicitly proved. You will need to use the definition of "mod n".)

IV. (6 points each) Do ONE of the following two problems.

1) Prove Cauchy's Theorem: Let G be a finite abelian group, p a prime which divides the order of G. Then G has an element of order p.

2) Prove Cayley's Theorem: Every group is isomorphic to a group of permutations.

V. (6 points each) Do FOUR of the following five problems.

1) a) Define what it means for a relation R to be an *equivalence relation* on a set S.

b) Let G be a group and H a subgroup of G. Define a relation R on the set of elements of G by $R(x, y)$ if and only if there is $h \in H$ with $x = hy$. Prove that R is an equivalence relation.

c) Find all the equivalence classes for the relation R above when G is the group S_3 of permutations of three elements and $H = \{\varepsilon, (1\ 2)\}$.

2) Let G be a group of order 21.

a) State Sylow's first and third theorems.

b) Find the possibilities for the numbers n_p of Sylow p-subgroups of G, for all prime divisors of 21.

c) Prove that G has a normal subgroup.

3) The aim of this problem is to show explicitly that $U(21)$ is isomorphic to $Z_2 \oplus Z_6$.

a) Write out $U(21)$.

b) Find subgroups H and K of G, of orders 2 and 6 respectively, such that $H \cap K = \emptyset$.

c) Show that $U(21) = HK$, that is, every element of $U(21)$ can be written as the product of an element of H and an element of K.

4) Recall that an *inner automorphism* of a group G is an automorphism φ_a of G defined by $\varphi_a(x) = axa^{-1}$, where a is any element of $g = G$. The set $\mathrm{Inn}(G)$ of inner automorphisms of G is a subgroup of $\mathrm{Aut}(G)$, the group of all automorphisms of G.

a) Prove $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

b) Prove that $\varphi_a = \varphi_{az}$ if and only if z is in the center of G.

Be careful! Remember that the operation in $\mathrm{Aut}(G)$ is function composition, and two functions are equal if they have the same effect on all elements of the group.

5) Let $G = Z_2 \oplus Z_8$ and $H = \langle (0, 4) \rangle$.

a) G/H is isomorphic to one of $Z_2 \oplus Z_2 \oplus Z_2$, $Z_2 \oplus Z_4$ or Z_8. Decide which, by calculating orders of elements in the quotient group.

b) Write down an explicit homomorphism from G to the correct group from a) whose kernel is H. (You do not need to prove that your function is a homomorphism.)

THE END