Abstract Algebra Math 3E03, Midterm 1, 20 October 2004

Write your solutions in the answer book provided, numbered clearly. If you do parts of questions in different parts of the booklet, please indicate this clearly.

Part I Answer ALL parts of the following three questions.

1) (15) Give careful precise definitions of the following words or phrases.

a) $(G, *)$ is a group
b) H is a subgroup of G
c) the function $f : A \to B$ is one-to-one and onto
d) the order $|a|$ of an element a of a group
e) the group G is cyclic

2) (12) Give examples for the following. Make sure that your example is stated in full, but you do not need to prove that it has the desired property.

a) a group
b) a set with a binary operation on it which is not a group
c) a function $f : \mathbf{Z} \to \mathbf{Z}$ which is one-to-one but not onto
d) an element of order 3 in a group with more than three elements

3) (9) Short answer questions.

a) State a subgroup test.

b) Let a be an element of a group G, with $|a| = n$. What is the condition on k for a^k to be a generator of $\langle a \rangle$? (You do not need to prove your answer.)

c) Write the permutation $\alpha = (a_1 a_2 \dots a_n)$ as a product of 2-cycles, and explain how to determine if α is even or odd.

Part II Answer TWO of the following four questions (6 each).

1) Let G be a group and define the *center* of G to be

$$Z(G) = \{a \in G : ax = xa \text{ for all } x \in G\}.$$

Prove that $Z(G)$ is a subgroup of G. Let G be the subgroup of $GL_2(\mathbf{R})$ given by:

$$G = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbf{R}, ac \neq 0\}.$$

Find $Z(G)$.

2) Let $G = \langle a \rangle$ be a cyclic group of order 63. List all the generators of G. Find all distinct subgroups of G and draw the subgroup lattice.

3) Prove by induction on n that $n^2 \leq 2^n$ for all $n \geq 4$.

4) Let G be a group of permutations of a set A. For $a \in A$ we define the stabiliser in G of a to be

$$\text{stab}(a) = \{\alpha \in G : \alpha(a) = a\}.$$

Prove that stab(a) is a subgroup of G. Write D_5, the group of symmetries of a regular pentagon, as the permutation group of the vertices (a subgroup of S_5). Find the stabiliser of the vertex labelled 1.