MATHEMATICS IS CONNECTED
JOINT ASSIGNMENT FOR 3EE, 3T, 3X

1. Let X be the solid torus $D^2 \times S^1$ where D^2 denotes the closed unit disc in $\mathbb{R}^2$. Compute the fundamental group of X using "first principles" (i.e. don't use Theorem 60.1 in Munkres - work directly using the definition of path-homotopies and the definition of fundamental group instead).

2. Let $T^2 = S^1 \times S^1$ be the 2-torus and $p \in T^2$. Let S^1 be the circle and $q \in S^1$. Let $T^2 \sqcup S^1$ be the disjoint union and let X be the quotient space obtained from $T^2 \sqcup S^1$ by gluing (identifying) the two points p and q. Give an explicit description of the fundamental group of X in terms of generators and relations. (You may assume the Seifert-van Kampen Theorem.)

3. (a) Prove that the one-point compactification of the complex plane $\mathbb{C}$ (with the usual topology) is homeomorphic to S^2.

   (b) Let $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C}$. Define a map $\phi : U_1 \setminus \{0\} \to U_2 \setminus \{0\}$ by $\phi(z) = z^{-1}$. Define an equivalence relation on the disjoint union $U_1 \sqcup U_2$ by: for $x \in U_1 \setminus \{0\}$ and $y \in U_2 \setminus \{0\}$,

   $$x \sim y \text{ if and only if } y = \phi(x).$$

   Let X denote the quotient space corresponding to $U_1 \sqcup U_2$ with this equivalence relation. Prove that X is homeomorphic to S^2 and hence the one-point compactification of $\mathbb{C}$. Notice that ϕ is a holomorphic function, where defined. Thus X is a topological space which is also equipped with a complex-analytic structure: for instance it makes sense to talk about *holomorphic functions on* X even though X is not a subset of the complex plane. Equipped with this extra structure, the space S^2 is often called the **Riemann sphere**. A meromorphic function f on a domain $\Omega \subseteq \mathbb{C}$ is, equivalently, a holomorphic function to the Riemann sphere.

4. In complex analysis, we talk about representing functions as power series or Laurent series. Alternatively, we can study the algebraic properties of the sets of all convergent power series or Laurent series. More generally, we can talk about the sets of *formal* series (where we do not impose the condition of convergence). Let

   $$\mathbb{C}[[X]] = \{f(X) = \sum_{i=1}^{\infty} f_i X^i : f_i \in \mathbb{C}\}, \text{ the set of formal power series over } \mathbb{C}, \text{ and}$$

   $$\mathbb{C}((X)) = \{f(X) = \sum_{i=M}^{\infty} f_i X^i : f_i \in \mathbb{C}, M \in \mathbb{Z}\} \text{ the set of formal Laurent series over } \mathbb{C}.$$

   For any $f(X) = \sum_{i=M}^{\infty} f_i X^i$, $g(X) = \sum_{i=N}^{\infty} g_i X^i$, let $P = \min\{M, N\}$ and set $f_i = 0$ for $P \leq i < M$, $g_i = 0$ for $P \leq i < M$. We define addition and multiplication on $\mathbb{C}((X))$ by:

   $$(f+g)(X) = \sum_{i=P}^{\infty} (f_i + g_i) X^i,$$

   $$(fg)(X) = \sum_{i=M+N}^{\infty} (\sum_{j=P}^{i} f_j g_{i-j}) X^i.$$

Prove the following.

(a) $\mathbb{C}((X))$ is a ring and $\mathbb{C}[[X]]$ is a subring.

(b) $f \in \mathbb{C}[[X]]$ is invertible if and only if $f_0 \neq 0$.

(c) $\mathbb{C}((X))$ is a field.

(d) $\mathbb{C}[[X]]$ has a unique maximal ideal I, and $\mathbb{C}[[X]]/I$ is isomorphic to $\mathbb{C}$.

5. (a) Suppose that k is a field and $S \subseteq k[x_1, \ldots, x_n]$. Define $V(S) = \{\bar{a} \in k^n : \text{for all } f \in S, f(\bar{a}) = 0\}$; we will call $V(S)$ the zero set for S or just a zero set.

Show that the collection of all sets $V(S)$ as S ranges over all subsets of $k[x_1, \ldots, x_n]$ is the set of closed sets of a topology on k^n. This topology is called the Zariski topology on k^n. Show that it is compact and Noetherian i.e. there is no infinite descending chain of closed sets. Hints: show that the union of two zero sets is a zero set, $V(S) = V(\langle S \rangle)$ and most importantly, $V(S) = V(S_0)$ for some finite $S_0 \subseteq S$.

(b) For a set $X \subseteq k^n$, let $I(X) = \{f \in k[x_1, \ldots, x_n] : \text{for all } \bar{a} \in X, f(\bar{a}) = 0\}$. Show that $I(X)$ is a radical ideal i.e. if $f^n \in I(X)$ for some n then $f \in I(X)$. Show that for any X, $V(I(X))$ is the closure of the set X in the Zariski topology.

(c) In a topology, a closed set is called irreducible if it cannot be written as the union of two proper, non-empty closed subsets. Show that $V(S)$ is irreducible in the Zariski topology iff the ideal generated by S is prime.