Priestley Chapter 13 Some solutions

13.1

$\gamma(0;2)$.

(i) $\displaystyle\int_\gamma \frac{z^3+5}{z-i}\,dz = 2\pi i\left(z^3+5\right)\Big|_{z=i}$ by Cauchy integral formula, as i inside γ.

$= 2\pi i(5-i)$.

(ii) $\displaystyle\int_\gamma \frac{1}{z^2+z+1}\,dz = \int_\gamma \frac{1}{(z-a_1)(z-a_2)}\,dz$ $\quad a_1 = \frac{1}{2}(-1+i\sqrt{3})$, $a_2 = \frac{1}{2}(-1-i\sqrt{3})$

$\displaystyle = \frac{-1}{i\sqrt{3}}\int_\gamma \frac{1}{z-a_1}\,dz + \frac{1}{i\sqrt{3}}\int_\gamma \frac{1}{z-a_2}\,dz$ partial fractions

$\displaystyle = \frac{-1}{i\sqrt{3}}\,2\pi i + \frac{1}{i\sqrt{3}}\,2\pi i$ by the fundamental integral, as a_1, a_2 both inside γ

$= \cancel{-\frac{4\pi}{\sqrt{3}}} = 0$.

(iii) $\displaystyle\int_\gamma \frac{\sin(z)}{z^2+1}\,dz = \frac{-1}{2i}\int_\gamma \frac{\sin(z)}{z+i}\,dz + \frac{1}{2i}\int_\gamma \frac{\sin(z)}{z-i}\,dz$

$\displaystyle = -\frac{1}{2i}\sin(z)\Big|_{z=-i} 2\pi i + \frac{1}{2i}\sin(z)\Big|_{z=i} 2\pi i$

$= 2\pi\sin(i)$.

$$\frac{13.9}{13.4}$$

13.9 $\quad \gamma(0;2)$.

(i) $\displaystyle\int_\gamma \frac{e^{z^2}}{(z-1)^3}\,dz = \frac{2\pi i}{2!} f^{(2)}(1)$ where $f(z)=e^{z^2}$ by Cauchy's formula for derivatives.

$= 6e\pi i$

(ii) $\displaystyle\int_\gamma \frac{\cos(z)}{z^n}\,dz = \frac{2\pi i}{(n-1)!}\left(\cos(z)\right)^{(n-1)}\Big|_{z=0}$ for $n\geq 1$.

$$= \begin{cases} \dfrac{2\pi i}{(n-1)!}(-1)^{\frac{n-1}{2}}, & n \text{ odd} \\ \cancel{2\pi i}\; 0, & n \text{ even} \end{cases}$$

(iii) $\displaystyle\int_\gamma \frac{1}{(z+1)^2(z^2+9)}\,dz = 2\pi i \left(\frac{1}{z^2+9}\right)'\Big|_{z=-1}$

$= \dfrac{\pi i}{25}$.

13.11 (i) a, b both inside γ - use partial fractions

one in, one out - integral is $\left(\frac{1}{z-b}\right)\Big|_{z=a} \cdot 2\pi i$

both out - $\int$ integral $= 0$.

(ii) a inside γ - integral is the first derivative of $(z-b)^2$ at $z=a$, $\times 2\pi i$

a outside γ - integral $= 0$.