Priestley Chapter 14 Some solutions

14.1 (i) $f(z) = \sin^2(z)$ around 0.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f^2(z) = \sum_{k=0}^{\infty} \left( \sum_{n=0}^{k} a_n a_{k-n} \right) z^k$.

Here $a_n = \begin{cases} 0, & \text{if } n \text{ even} \\ \frac{(-1)^{n+1/2}}{n!}, & \text{if } n \text{ odd} \end{cases}$

$k$ odd $\Rightarrow$ either $n$ or $k-n$ even, so $\sum_{n=0}^{k} a_n a_{k-n} = 0$.

$k$ even: $\sum_{n=0}^{k} a_n a_{k-n} = \sum_{n \text{ odd}} a_n a_{k-n} = \sum_{m=0}^{\frac{k}{2}-1} a_{2m+1} a_{k-(2m+1)}$

Simplifying this requires some effort, and it is probably easier to calculate the power series directly from the derivatives:

$$\sin^2(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{n+1}}{(2n)!} z^{2n}.$$

(ii) $f(z) = \frac{1}{1+z}$ around $a = i$.

$$= \frac{1}{1+i} \sum_{n=0}^{\infty} \left( \frac{-1}{1+i} \right)^n (z-i)^n,$$ converges for $|z-i| < \sqrt{2}$.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^{n+1}} (z-i)^n$$

(iii) $$f(z) = e^z = e^{1+z-1} = e^1 e^{z-1}$$

$$f(z) = e \sum_{n=0}^{\infty} \frac{1}{n!} (z-1)^n$$ converges for all $z$.

$$= \sum_{n=0}^{\infty} \frac{e}{n!} (z-1)^n$$