Complex Analysis - Practice problems for final.

Mostly theory

1) State and prove Cauchy's Integral Formula.

2) State and prove Liouville's theorem.

3) Use Liouville's theorem to prove the fundamental theorem of algebra.

4) State Cauchy's formula for derivatives

5) State the Cauchy-Riemann equations

6) State Cauchy's ~~formula~~ theorem (be careful about the hypotheses).

7) State Rouché's theorem. Find the number of ~~zeroes of~~ solutions of the equation $e^z - 4z^n = 0$ inside the unit circle

8) State the integral form of the coefficients of the Laurent series expansion for a function.

9) State and prove Cauchy's residue theorem.

Mostly practice

1) Solve the equations: $\cos(z) - 1 = 0$, $z^4 = 16$

2) Let f be a function which is holomorphic on a region G. Suppose that $|f(z)^2 - 1| < 1$ for all $z \in G$. Show that either $\operatorname{Re} f(z) > 0$ for all $z \in G$ or $\operatorname{Re}(f(z)) < 0$ for all $z \in G$.

3) Calculate the fundamental integral using contour integration. That is, show
$$\int_{\gamma(a;1)} (z-a)^n \, dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

4) Evaluate the following integrals:

$\int_\gamma \frac{1}{z} \, dz$; $\gamma(t) = \cos(t) + 2i\sin(t)$; $0 \le t \le 2\pi$

$\int_\gamma \frac{1}{z^2} \, dz$; γ as above

$\int_\gamma \frac{e^z}{z} \, dz$; $\gamma(t) = 2 + e^{it}$, $0 \le t \le 2\pi$

$\int_\gamma \frac{1}{z^2 - 1} \, dz$; $\gamma(1;1)$ circle center 1, radius 1

$\int_\gamma \frac{1}{z} \, dz$; γ any curve which can be deformed to the unit circle (without crossing 0)

5) Evaluate the following integrals:

$$\int_{|z|=\frac{1}{2}} \frac{1}{(1-z)^3}\,dz \quad , \quad \int_{|z+1|=\frac{1}{2}} \frac{1}{(1-z)^3}\,dz$$

$$\int_{|z-1|=\frac{1}{2}} \frac{1}{(1-z)^3}\,dz$$

6) Evaluate the following integrals

$$\int_{\gamma(0;2)} \frac{z^2}{z-1}\,dz \quad , \quad \int_{\gamma(0;1)} \frac{e^z}{z^2}\,dz$$

7) Suppose f is holomorphic inside and on the closed curve γ. Suppose $f=0$ on γ. Show that $f(z)=0$ for all z inside γ.

8) Evaluate the following integrals, when γ is the circle center 0, radius 2.

$$\int_\gamma \frac{1}{z^2-1}\,dz \quad , \quad \int_\gamma \frac{1}{z^2+z+1}\,dz \quad , \quad \int_\gamma \frac{1}{z^2-8}\,dz$$

9) Compute the Taylor series (or just the first four terms) of the following functions about the given point: $\frac{\sin(z)}{z}$; $a=1$.

$z^2 e^z$; $a=0$; $\qquad e^z \sin(z)$; $a=0$.

10) Find the Laurent series expansion (~~the~~ a few terms) around $a=0$ for the following functions

$$\sin\left(\frac{1}{z}\right) \; ; \quad \frac{1}{z(z+1)} \; ; \quad \frac{z}{z+1} \; ; \quad \frac{e^z}{z^2}$$

11) Which of the following functions have removable singularities at the given point?

$$\frac{\cos(z-1)}{z^2} \text{ at } a=0 \; ; \quad \frac{z}{z-1} \text{ at } a=1 \; ;$$

$$\frac{f(z)}{(z-a)^n} \; , \text{ if } f \text{ has a zero at } a \text{ of order } n.$$

12) Find the residues of the given function at the relevant points.

$$\frac{e^z-1}{\sin(z)} \; ; \quad \frac{1}{e^z-1} \; ; \quad \frac{z+2}{z^2-2z} \; ; \quad \frac{1+e^z}{z^4} \; ;$$

$$\frac{e^z}{(z^2-1)^2} \; ; \quad \frac{e^{z^2}}{z-1} \; ; \quad \frac{e^{z^2}}{(z-1)^2} \; ; \quad \left(\frac{\cos(z)-1}{z}\right)^2$$

$$\frac{z^2}{z^4-1} \; ; \quad \frac{1}{z^3(z+4)} \; ; \quad \frac{1}{z^2+2z+1} \; ; \quad \frac{1}{z^3-3} \, .$$

13) Evaluate the integral $\int_\gamma \frac{1}{z^4+1}\,dz$, where γ is the line segment from -2 to 2 on the real axis, followed by the semicircle from 2 to -2 traversed counterclockwise.

14) Evaluate $\int_\gamma \frac{1+z}{1-\cos z}\, dz$, where γ is circle centre 0 radius 7.

15) Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2-2x+4}\, dx$.

16) Evaluate $\int_0^{\infty} \frac{\cos(2x)}{1+x^4}\, dx$