Solution to assignment 2.

1. proof: Let $f(z) = f(x, y) = u(x, y) + i v(x, y)$,

where $z = x + iy$. Since f is holomorphic on a region G, ~~R~~. Cauchy - Riemann equations hold on G.

Therefore, we have

$$u_x = v_y, \quad u_y = -v_x.$$

By assumption that $\operatorname{Re} f = u(x, y)$ is a constant on G.

So $u_x = 0$ and $u_y = 0$ on G.

Combining the Cauchy Riemann equations, we get

$$v_x = v_y = 0 \text{ on } G.$$

Thus, $f'(z) = u_x + i v_x = 0$ on G.

So f is constant on G.

2. $$f(z) = \frac{1}{1-z} = \frac{1}{2-(z+1)} = \frac{1}{2} \cdot \frac{1}{1 - \frac{z+1}{2}}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z+1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (z+1)^n$$

The series converge absolutely on the set $\{z \in \mathbb{C} : |z+1| < 2\}$.

Again, $f(z) = \dfrac{1}{1-z} = \dfrac{1}{(1-i)-(z-i)}$

$$= \frac{1}{1-i} \cdot \frac{1}{1 - \frac{z-i}{1-i}}$$

$$= \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{1-i}\right)^{n+1} (z-i)^n$$

The series converges absolutely on the set

$$\left\{ z \in \mathbb{C} : \left|\frac{z-i}{1-i}\right| < 1 \right\} = \left\{ z \in \mathbb{C} : |z-i| < \sqrt{2} \right\}$$

$$g(z) = \frac{1}{z(z+2)} = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{z+1-1} - \frac{1}{1+z+1}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{1-(z+1)} - \frac{1}{1-[-(z+1)]}\right)$$

$$= \frac{1}{2}\left(-\sum_{n=0}^{\infty} (z+1)^n - \sum_{n=0}^{\infty} (-1)^n (z+1)^n\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}(-1)^{n+1}(z+1)^n - \sum_{n=0}^{\infty}(z+1)^n\right)$$

The series converges absolutely on the set $\{z \in \mathbb{C} : |z+1| < 1\}$

Again, $g(z) = \frac{1}{z(z+2)} = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right)$

$$= \frac{1}{2}\left(\frac{1}{i+z-i} - \frac{1}{2+i+z-i}\right)$$

$$= \frac{1}{2}\left(\frac{1}{i}\cdot\frac{1}{1-(-\frac{z-i}{i})} - \frac{1}{2+i}\cdot\frac{1}{1-(-\frac{z-i}{2+i})}\right)$$

$$= \frac{1}{2}\left(\frac{1}{i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{i}\right)^n - \frac{1}{2+i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2+i}\right)^n\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{i^{n+1}}(z-i)^n - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2+i)^{n+1}}(z-i)^n\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{i^n}{i}(z-i)^n - \sum_{n=0}^{\infty}\frac{1}{2+i}\left(-\frac{1}{2+i}\right)^n(z-i)^n\right)$$

The series converges absolutely on the set

$$\left\{z \in \mathbb{C} : \left|\frac{z-i}{i}\right| < 1\right\} \cap \left\{z \in \mathbb{C} : \left|\frac{z-i}{2+i}\right| < 1\right\}$$

$$= \{z \in \mathbb{C} : |z-i| < 1\} \cap \{z \in \mathbb{C} \ |z-i| < \sqrt{5}\}$$

$$= \{z \in \mathbb{C} : |z-i| < 1\}$$

3. (1) $f(z) = e^{\frac{1}{z}}$ is holomorphic on the set $\{z \in \mathbb{C} : z \neq 0\}$

and $f'(z) = e^{\frac{1}{z}} \cdot \left(-\frac{1}{z^2}\right) = -\frac{1}{z^2} e^{\frac{1}{z}}$

(2) $f(z) = e^{\frac{1}{1-az}}$, $a \in \mathbb{C}$ is holomorphic on the set

$$\{z \in \mathbb{C} : 1 - az \neq 0\} = \{z \in \mathbb{C} : z \neq \frac{1}{a}\}$$

and $f'(z) = e^{\frac{1}{1-az}} \cdot \frac{a}{(1-az)^2} = \frac{a}{(1-az)^2} e^{\frac{1}{1-az}}$

(3) $f(z) = \frac{\sin z}{z}$ is holomorphic on the set $\{z \in \mathbb{C} : z \neq 0\}$

and $f'(z) = \frac{z \cos z - \sin z}{z^2}$

4. (1) Since $\cos z = -1$ implies that $z = (2K+1)\pi$, $K \in \mathbb{Z}$.

and $\cosh z = \cos iz$ by the definitions of $\cos z$ and $\cosh z$.

Thus $\cosh z = -1$ is equivalent to $\cos iz = -1$.

i.e. we have $iz = (2k+1)\pi$, $k \in \mathbb{Z}$.

So $\cosh z = -1$ implies that $z = -i(2k+1)\pi$, $k \in \mathbb{Z}$.

(2) $$\cos^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 = \frac{e^{2iz}+2+e^{-2iz}}{4}$$

$$= \frac{\cos 2z + 1}{2}$$

So $\cos^2 z = 4 \iff \cos 2z = 7$ ①

Let $z = x+iy$, $x, y \in \mathbb{R}$, then

$$\cos 2z = \frac{1}{2}(e^{i2z} + e^{-i2z})$$

$$= \frac{1}{2}(e^{-2y}e^{i2x} + e^{2y}e^{-i2x})$$

$$= \frac{1}{2}\cos 2x\,(e^{2y}+e^{-2y}) + i\cdot\frac{1}{2}\sin 2x\,(e^{-2y}-e^{2y}) \quad ②$$

By ① and ②, we get

$$\begin{cases} \frac{1}{2}\cos 2x\,(e^{2y}+e^{-2y}) = 7 \\ \frac{1}{2}\sin 2x\,(e^{-2y}-e^{2y}) = 0 \end{cases}$$

$\Updownarrow$ Let $x' = 2x$, $y' = 2y$

$$\begin{cases} \cos x'\,(e^{y'}+e^{-y'}) = 14 \quad ③ \\ \sin x'\,(e^{-y'}-e^{y}) = 0 \Rightarrow x' = k\pi,\ k\in\mathbb{Z}.\ \text{or}\ y'=0 \end{cases}$$

If $y'=0$, we obtain that $\cos x' = 7$ from the equation

$\cos x'(e^{y'}+e^{-y'}) = 14$, which is impossible. So $y' \neq 0$.

and $x' = k\pi$, $k \in \mathbb{Z}$.

If $x' = 2k\pi$, $\cos x' = 1$. it follows from ③ that

$$e^{y'} + e^{-y'} = 14 \iff e^{2y'} - 14e^{y'} + 1 = 0$$

So $$e^{y'} = \frac{14 \pm \sqrt{14^2-4}}{2} = \frac{14 \pm \sqrt{194}}{2}$$

i.e. $$y' = \ln \frac{14 \pm \sqrt{194}}{2}$$

In this case, $z = x + iy = \frac{1}{2}(x' + iy')$

$$= k\pi + i \cdot \frac{1}{2} \ln \frac{14 \pm \sqrt{194}}{2}, \quad k \in \mathbb{Z}$$

If $x' = (2k+1)\pi$, $\cos x' = -1$, it follows from ③ that

$e^{y'} + e^{-y'} = -14$, which is impossible since

$e^{y'}$ and $e^{-y'}$ are both positive.

Therefore, the solution to $\cos^2 z = 4$ is

$$z = k\pi + \frac{i}{2} \ln \frac{14 \pm \sqrt{194}}{2}, \quad k \in \mathbb{Z}.$$

(3) $\tan z = i \iff \dfrac{\sin z}{\cos z} = i \iff \sin z = i\cos z$ & $\cos z \neq 0$

By the definition of $\sin z$ and $\cos z$, we get

$$\sin z = i\cos z \iff \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{i}{2}(e^{iz} + e^{-iz})$$

$$\iff ie^{iz} = 0 \iff e^{iz} = 0$$

~~Let z=~~ So no solution to the equation $\tan z = i$