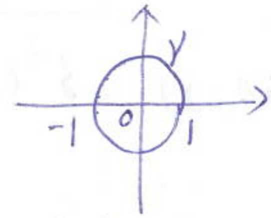
Homework 4:

1. (a) $$\int_{\gamma(0;1)} \frac{e^z}{z^2}\, dz$$



Cauchy's formula for the first derivative $\rightarrow$ $2\pi i \left. \dfrac{d}{dz} e^z \right|_{z=0}$

$= 2\pi i$

The fct e^z is holomorphic in the whole complex plane, so it's holomorphic inside and on $\gamma(0;1)$. Since $0 \in I(\gamma(0;1))$, then Cauchy's formula for the first derivative can be used.

(b) $$\int_{\gamma(1;2)} \frac{1}{z^2 - 2i}\, dz$$

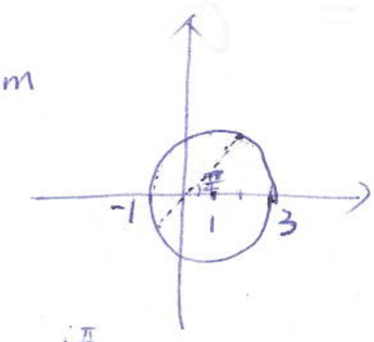
$$= \int_{\gamma(1;2)} \frac{1}{\sqrt{2}\,(e^{i\frac{\pi}{4}} - e^{i\frac{5\pi}{4}})} \left(\frac{1}{z - \sqrt{2}e^{i\frac{\pi}{4}}} - \frac{1}{z - \sqrt{2}e^{i\frac{5\pi}{4}}} \right)$$

$$= \frac{1}{2(1+i)} \left[\int_{\gamma(1;2)} \frac{1}{z - \sqrt{2}e^{i\frac{\pi}{4}}}\, dz - \int_{\gamma(1;2)} \frac{1}{z - \sqrt{2}e^{i\frac{5\pi}{4}}}\, dz \right]$$

Cauchy's integral formula

Cauchy's Theorem

$$= \frac{1}{2(1+i)} \left[2\pi i - 0 \right]$$



$$= \frac{\pi i}{1+i}$$

$$= \frac{\pi}{2}(1+i)$$

$|\sqrt{2}e^{i\frac{\pi}{4}} - 1| = |i| = 1 < 2$

$\Rightarrow \sqrt{2}e^{i\frac{\pi}{4}} \in I(\gamma(1;2))$

$|\sqrt{2}e^{i\frac{5\pi}{4}} - 1| = |-2-i| = \sqrt{5} > 2$

$\Rightarrow \sqrt{2}e^{i\frac{5\pi}{4}} \in O(\gamma(1;2))$

(c) $\displaystyle\int_{\gamma(0;2)} \frac{1}{z^2(z^2+16)}\, dz$

$$= \frac{1}{16}\int_{\gamma(0;2)} \left(\frac{1}{z^2} - \frac{1}{z^2+16}\right) dz$$

$$= \frac{1}{16}\left[\int_{\gamma(0;2)} \frac{1}{z^2}\, dz - \int_{\gamma(0;2)} \frac{1}{(z+4i)(z-4i)}\, dz\right]$$

$$= \frac{1}{16}\left[\int_{\gamma(0;2)} \frac{1}{z^2}\, dz - \frac{1}{8i}\int_{\gamma(0;2)} \frac{1}{z-4i}\, dz + \frac{1}{8i}\int_{\gamma(0;2)} \frac{1}{z+4i}\, dz\right]$$

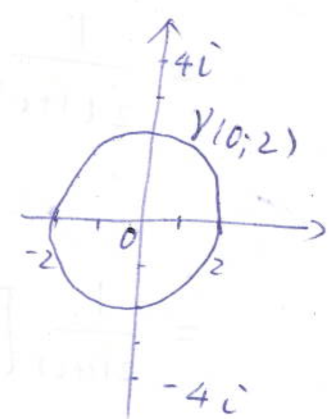
(The fundamental integral) or Cauchy's formula for the first derivative

Cauchy's theorem

Cauchy's theorem

$$= \frac{1}{16}\left[2\pi i \frac{d}{dz} 1\Big|_{z=0} - 0 + 0\right]$$

$$= 0$$



2. proof: Since f is holomorphic on the entire complex plane, then we can let the Taylor expansion of f (around $a=0$) be $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for $z \in \mathbb{C}$. Then

$$\frac{f(z)}{z} = \frac{c_0}{z} + c_1 + c_2 z + c_3 z^2 + \cdots \quad (1)$$

By assumption, $\lim_{z \to \infty} \frac{f(z)}{z} = 0$. We get

⇒ #: defend this assertion

$c_i = 0$ for $i \geq 1$ by formula (1). This shows that $f(z) = c_0$.

3. By Cauchy's formula for the first derivative, we have

$$f'(0) = \frac{1}{2\pi i} \int_{\gamma(0;1)} \frac{f(z)}{z^2} \, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{2i\theta}} \, i e^{i\theta} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta}} \, d\theta$$

Thus $|f'(0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta}} \, d\theta \right|$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta})|}{|e^{i\theta}|} \, d\theta$$

Since $|f(z)| \leq 1$ for $z \in D(0;1)$ $\longrightarrow \leq \frac{1}{2\pi} \int_0^{2\pi} d\theta$

$$= 1 .$$

For example, we can take function $f(z) = z$.

Then $f(z) = z$ satisfies all conditions in our problem.

and $f'(z) \equiv 1$. Therefore, $f'(0) = 1$, which achieves the upper bound we gave in the above.

4. (a) Let $f(z) = e^z$, then $f^{(n)}(z) = e^z$.

By Taylor's theorem, the Taylor series for e^z around $a = 1$ is

$$e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (z-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{e}{n!} (z-1)^n$$

(b) ~~e^z~~ The Taylor series for the function e^{z^2} is

$$e^{z^2} = \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$$

(using the expansion of e^z around $a=0$)

(c) Let $g(z) = \sin z$, then $g^{(n)}(z) = \sin(z + \frac{n\pi}{2})$

Therefore, the Taylor series for the function $g(z) = \sin z$ around $a=1$ is

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(1)}{n!} (z-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{\sin(1+\frac{n\pi}{2})}{n!} (z-1)^n$$

Let $h(z) = \frac{1}{z}$, then $h(z) = \frac{1}{z-1+1} = \frac{1}{1-[-(z-1)]} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$

Thus the Taylor series for $\frac{\sin z}{z}$ around $z=1$ is

$$\frac{\sin z}{z} = g(z)\,h(z)$$

$$= \sum_{n=0}^{\infty} \frac{\sin(1+\frac{n\pi}{2})}{n!}(z-1)^n \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

or, moreover, the Taylor series of $\frac{\sin z}{z}$ is in the following form.

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} C_n (z-1)^n, \quad \text{where}$$

$$C_n = \sum_{r=0}^{n} a_r b_{n-r}, \quad a_r = \frac{\sin(1+\frac{r\pi}{2})}{r!}, \quad b_r = (-1)^r$$

(d) $$\frac{1}{(z-1)(z-i)} = \frac{1}{1-i}\left(\frac{1}{z-1} - \frac{1}{z-i}\right)$$

$$= \frac{1}{1-i}\left(-\frac{1}{1-z} + \frac{1}{i(1-\frac{z}{i})}\right)$$

$$= \frac{1}{1-i}\left(\frac{1}{i}\sum_{n=0}^{\infty}\left(\frac{1}{i}\right)^n z^n - \sum_{n=0}^{\infty} z^n\right)$$

$$= \frac{1}{1-i}\left(\sum_{n=0}^{\infty}\left[(-i)^{n+1} - 1\right] z^n\right)$$