Homework 5: Zeroes and singularities

due 8:30 22 March 2011

1) Let $f(z) = \dfrac{\pi z(1-z^2)}{\sin(\pi z)}$.

(a) Find all zeroes of f and their orders.

(b) Find all singularities of f and classify them.

(c) Discuss the Laurent expansion of f around 0, and calculate the first few non-zero coefficients.

2) Consider the function $f(z) = \dfrac{1}{z(1-z)(2-z)}$. Find Laurent series expansions for f as follows (give the series as a product, and give the coefficients of the principal part):

(a) around 0, convergent in the annulus $\{z : 0 < |z| < 1\}$,

(b) around 1, convergent in the annulus $\{z : 0 < |z-1| < 1\}$.

3) Use Taylor series to prove the following complex version of L'Hopital's rule: Let $f(z)$ and $g(z)$ be holomorphic, each with a zero of order k at a. Then $f(z)/g(z)$ has a removable singularity at a and

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{f^{(k)}(a)}{g^{(k)}(a)} \ .$$

4) Find the residues at all singularities of the following functions:

(a) $f(z) = \dfrac{z^2}{\sin^2(z)}$,

(b) $f(z) = \dfrac{z^2-1}{(z^2+1)^2}$.