Math 3X 2010-11. Solutions to Midterm 1

1) (i) Solve the equation $1 - z^2 + z^4 = 0$.

~~Expone~~ Solve as a quadratic in z^2, then take the square root. Remember we expect 4 solutions:

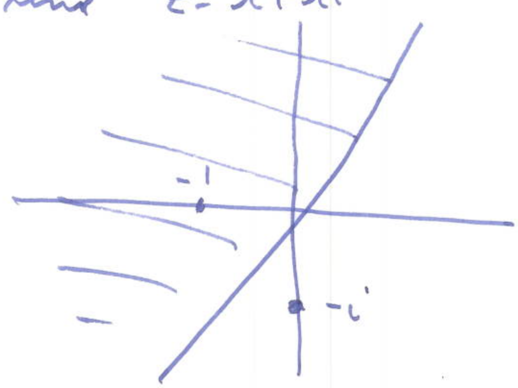
$$z = e^{i\pi/6},\ e^{i5\pi/6},\ e^{i7\pi/6},\ e^{i11\pi/6}$$

<u>or</u> multiply both sides of the equation by $1+z^2$: solve $1+z^6 = 0$, $1+z^2 \neq 0$.

$\boxed{\begin{array}{l}\text{(ii) } \cos(z) = i. \\ \text{Write } z = x+iy, \\ \text{use trig identities.} \\ z = \frac{\pi}{2} + 2\pi k + i\ln(-1+\sqrt{2}) \\ z = \frac{\pi}{2} + (2k+1)\pi + i\ln(1+\sqrt{2})\end{array}}$

2) $S = \{ z \in \mathbb{C} : |z+1| < |z+i| \}$.

Since $|z+1| = |z+i|$ is the set of points equidistant from the points -1 and $-i$, this is a line $x = y$. $z = -1$ is in the set S, so S is the open half-plane above the line $z = x + xi$



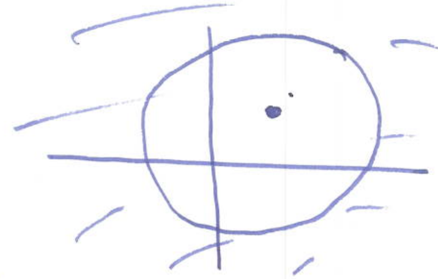
$f(S) = \{ f(z) : |z+1| < |z+i| \}$,

where $f(z) = \dfrac{1}{z+1}$.

Write $w = \dfrac{1}{z+1}$, rewrite $z = \dfrac{1-w}{w}$.

Substitute in $|z+1| = |z+i|$ and manipulate to get

$|w - \frac{1}{2}(1+i)| = \sqrt{2}$, which is a circle, centre $\frac{1}{2}(1+i)$, radius 1.



Since $f(-1) = \frac{1}{0} = \infty$, $f(S)$ is the outside of this circle.

3) $D = \{ re^{i\theta} : r < 1,\ 0 \le \theta \le 2\pi \}$

$f_1(D) = \{ \sqrt{r} e^{i\theta/2} : r < 1,\ 0 \le \theta \le 2\pi \}$

$= \{ R e^{i\varphi} : R < 1,\ 0 \le \varphi \le \pi \}$

$f_2(D) = \{ \sqrt{r} e^{i(\theta/2 + \pi)} : r < 1,\ 0 \le \theta \le 2\pi \}$

$= \{ \sqrt{r} e^{i(\theta/2 + \pi)} : r < 1,\ \pi \le \frac{\theta}{2} + \pi \le 2\pi \}$

4) $f(z) = \frac{1}{1 - e^z}$

(i) holomorphic for $1 - e^z \neq 0$.

Solve $e^z = 1$, get $z = 2\pi i k$

Holomorphic on $\mathbb{C} \setminus \{ 2\pi k i : k \in \mathbb{Z} \}$

(ii) $\frac{1}{1-w}$ converges for $|w| < 1$, so the composed power series converges for $|z| < 1$, which is $R(z) < 0$

(iii) $\frac{1}{1 - e^z} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{1}{k!} z^k \right)^n$ — rather hard to express as a power series!