Complex Analysis Midterm 2 Solutions

#1/16

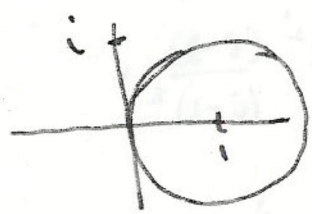
1) $$\int_\gamma \frac{e^z}{(z-1)(z-i)^2}\,dz$$

(i) $\gamma(0;1/2)$

The function is holomorphic inside and on γ, so by Cauchy's theorem,

$$\int_\gamma \frac{e^z}{(z-1)(z-i)^2}\,dz = 0.$$

(ii) $\gamma(1;1)$



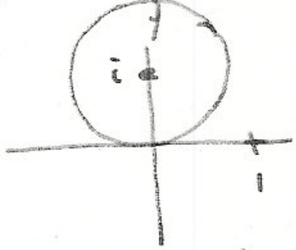
The function has a pole at $z=1$ inside the curve; but $\frac{e^z}{(z-i)^2}$ is holomorphic inside and on γ.

By Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_{\gamma(1;1)} \frac{e^z/(z-i)^2}{z-1}\,dz = \left.\frac{e^z}{(z-i)^2}\right|_{z=1} = \frac{e^1}{(1-i)^2}$$

Thus $$\int_{\gamma(1;1)} \frac{e^z}{(z-1)(z-i)^2}\,dz = \frac{2\pi i e}{(1-i)^2}.$$

(iii) $\int_{\gamma(i;1)}$



The function $g(z)=\frac{e^z}{z-1}$ ~~is~~ is holomorphic inside and on $\gamma(i;1)$. By Cauchy's integral formula for derivatives,

$$g'(i) = \frac{1!}{2\pi i}\int_{\gamma(i;1)} \frac{g(z)}{(z-i)^2}\,dz$$

$$g(z)=\frac{e^z}{z-1} = e^z(z-1)^{-1}$$

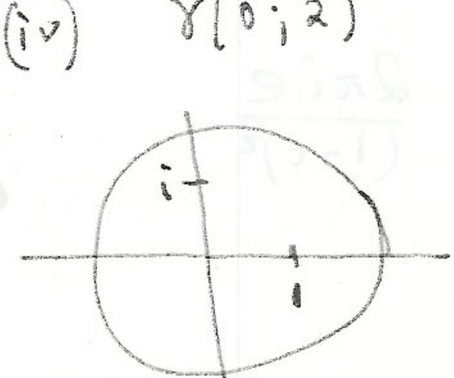
$$g'(z) = e^z(-1)(z-1)^{-2} + e^z(z-1)^{-1}.$$

$$g'(i) = e^i\left(\frac{-1}{(i-1)^2}+\frac{1}{i-1}\right) = e^i\left(\frac{-1+i-1}{(i-1)^2}\right)$$

$$= e^i\,\frac{(i-2)}{(i-1)^2}.$$

Thus $$\int_{\gamma(i,1)}\frac{e^z}{(z-1)(z-i)^2}\,dz = 2\pi i\, e^i\,\frac{(i-2)}{(i-1)^2}.$$

(iv) $\gamma(0;2)$



$$\frac{1}{(z-1)(z-i)^2} = \frac{-1/2i}{z-1} + \frac{\frac{1}{2i}z+\frac{1}{2i}-1}{(z-i)^2}$$

$$\int_{\gamma(0;2)} \frac{e^z}{(z-1)(z-i)^2}\,dz = \int_{\gamma(0,2)} \frac{-\frac{1}{2}i e^z}{(z-1)}\,dz + \int_{\gamma(0;2)} \frac{\left(\frac{1}{2i}z+\frac{1}{2i}-1\right)e^z}{(z-i)^2}\,dz$$

$$\cancel{= -\frac{1}{2}i \frac{2\pi i e}{(1-i)^2}}$$

$$= 2\pi i \left(-\frac{1}{2} i e^z\right)\Big|_{z=1} + \frac{2\pi i}{1!}\left(\left(\frac{1}{2i}z+\frac{1}{2i}-1\right)e^z\right)'\Big|_{z=i}$$

$$= \frac{2\pi e^1}{2} + 2\pi i\left(\frac{1}{i}-\frac{1}{2}\right)e^i.$$

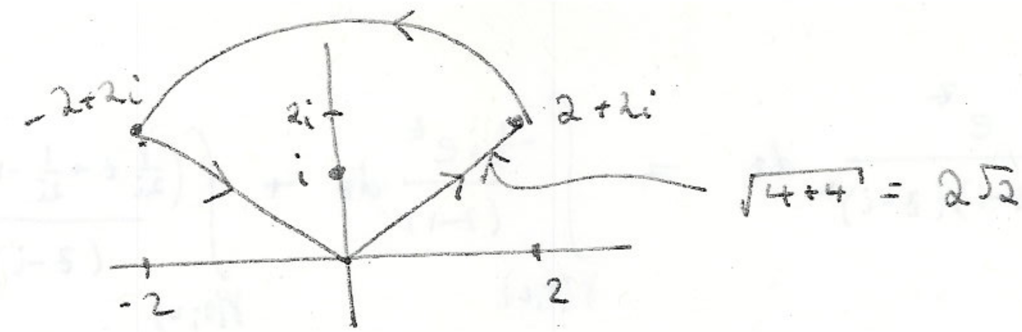
#2/6

2) $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, for all z.

$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, for $|z|<1$.

$\frac{e^z}{1-z} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} 1.\frac{1}{m!}\right) z^n$, converges for $|z|<1$.

#3/13 3)



(i) $\int_\gamma \frac{1}{z-i}\,dz = 2\pi i$ by the deformation theorem and the fundamental integral.

(ii) $[0,\ 2+2i] : \gamma(t) = (1-t)0 + t(2+2i)\ ,\ 0 \le t \le 1$

circular arc, radius $2\sqrt{2}$: $\gamma(t) = 2\sqrt{2}\,e^{it}$

$\frac{\pi}{4} \le t \le \frac{3\pi}{4}$

$[-2+2i,\ 0] : \gamma(t) = (1-t)(-2+2i) + t(0),\ 0 \le t \le 1$

(iii) $\int_\gamma |z|^2\,dz = \int_{\gamma_1} |z|^2\,dz + \int_{\gamma_2} |z|^2\,dz + \int_{\gamma_3} |z|^2\,dz$

$$\int_{\gamma_1} |z|^2\,dz = \int_{t=0}^{1} \left((2t)^2 + (2t)^2\right)(2+2i)\,dt = \int_0^1 8t^2 (2+2i)\,dt = \frac{8}{3}(2+2i)$$

$$\int_{\gamma_3} |z|^2\,dz = \int_{t=0}^{1} (1-t)^2\, 8\,(2-2i)\,dt = \frac{8}{3}(2-2i)$$

5

$$\int_{\gamma_2} |z|^2 dz = \int_{t=\pi/4}^{3\pi/4} 2\sqrt{2}\, 8\; i\, e^{it}\, dt = 8\left(e^{3\pi i/4} - e^{i\pi/4}\right)$$

$$= -16/\sqrt{2} \cdot 2\sqrt{2}$$

$$= -32$$