SOLUTION TO QUESTION 2.R.2 (MATH3X03)

Appologies for the messy presentation of the solution during the tutorial. I did not realize that a '**region**' is an '**open connected subsect** of $\mathbb{C}$' already. (see Page 49 of the book). Here is a full solution of the exercise if you are worrying about it before the exam.

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2.R.2. Let B be the set of $z \in A$ such that $f(z) = 0$. Then since B is a collection of isolated points in A, $A \backslash B$ is open and connected as well.

Consider the function $h(z) = f(z)/g(z)$, which is defined on $A \backslash B$, since $f(z) \neq 0$ on $A \backslash B$. It is clear that $h(z)$ is analytic on $A \backslash B$, and $|h(z)| = 1$ for all $z \in A \backslash B$. (Starting from here, a quick way to see that $h(z)$ is a constant on $A \backslash B$ is by using Exercise 2.R.7. Pick any point $z_0 \in A \backslash B$; we always have $|h(z)| \leq |h(z_0)|$ for all $z \in A \backslash B$, so Exercise 2.R.7 says that $h(z)$ is a constant on $A \backslash B$. But one can also give an independent proof as follows:)

Since $A \backslash B$ is open connected, pick some point $z_0 \in A \backslash B$, and consider the set

$$D = \{w \in A \backslash B \mid h(w) = h(z_0)\}.$$

we show that D is open and closed in $A \backslash B$, so then D has to be $A \backslash B$ itself, since D is not empty and $A \backslash B$ is connected.

For any $w \in D$, there exists some disc $D(w, r)$ contained in $A \backslash B$, since $A \backslash B$ is open. The function $h(z)$ is then analytic on the open connected and bounded smaller disc $D(w, r/2)$, and continuous on $\mathrm{cl}(D(w, r/2)) \subseteq A \backslash B$. So by the Maximum Modulus Theorem, since $|h(z)|$ attains its maximum 1 insider $D(w, r/2)$, $h(z)$ must be a constant on $D(w, r/2)$. This means $h(z) = h(w) = h(z_0)$ for all $z \in D(w, r/2)$. So D is open.

By the same reason, if $w \notin D$ and $w \in A \backslash B$, then there is a tiny disc containing w on which $h(z) = h(w) \neq h(z_0)$. So the complement of D in $A \backslash B$ is open. Thus D is closed.

So we conclude that $h(z)$ is a constant on $A \backslash B$. Thus $h(z) = e^{i\theta}$ for some constant $\theta \in [0, 2\pi)$. Therefore, $f(z) = e^{i\theta} g(z)$ on $A \backslash B$. But on B, $|f| = |g|$ is still true; so $f = 0$ on B means $g = 0$ as well, and naturally $f = e^{i\theta} g$ is also true on B.

Finally, we have shown that $f(z) = e^{i\theta} g(z)$ on A for some constant $\theta \in [0, 2\pi)$.

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