SOLUTIONS FOR ASSIGNMENT 1 (MATH3X03)

1.1.10. (4 pts)

(a) (Note that ϕ_z is indeed a linear map, so one can talk about its matrix.) We have the following (by Definition 1.1.1 in the textbook):

$$\phi_z((1,0)) = (x,y)(1,0) = (x,y) = x(1,0) + y(0,1),$$
$$\phi_z((0,1)) = (x,y)(0,1) = (-y,x) = -y(1,0) + x(0,1)$$

witten into the matrix form as:

$$\phi_z\big[(1,0),(0,1)\big] = \big[(1,0),(0,1)\big]\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Hence the matrix of ϕ_z is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

(b) (It is not very clear what the question is asking for. One may just say that $\phi_{z_1 z_2} = \phi_{z_1}\phi_{z_2}$ because of the associativity of the complex multiplication, but this might just what the question is asking for. Otherwise, see the following...)

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$. So the matrix corresponding to $\phi_{z_1z_2}$ is

$$\begin{pmatrix} x_1x_2 - y_1y_2 & -x_1y_2 - x_2y_1 \\ x_1y_2 + x_2y_1 & x_1x_2 - y_1y_2 \end{pmatrix},$$

and the matrix corresponding to $\phi_{z_1} \circ \phi_{z_2}$ is

$$\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 & -x_1y_2 - x_2y_1 \\ x_2y_1 + x_1y_2 & x_1x_2 - y_1y_2 \end{pmatrix}$$

So we conclude that there are equal.

1.1.18. (4 pts)

(a)

$$(1-i)^{-1} = \frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{1+1} = \frac{1}{2} + i\frac{1}{2}.$$

(b)

$$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1-1+2i}{1+1} = i.$$

Date: 13 Jan, 2012. Tatally 40 points.

1.2.8. (2 pts)

$$\left|\frac{(2-3i)^2}{(8+6i)^2}\right| = \frac{|2-3i|^2}{|8+6i|^2} = \frac{2^2+3^2}{8^2+6^2} = \frac{13}{100}.$$

1.2.24. (6 pts)

For any z satisfying $|z| < 2$, we can write it as $re^{i\theta}$ where $0 \le r < 2$ and $0 \le \theta < 2\pi$. Then

$$\begin{aligned}\mathrm{Re}(iz^3+1) &= \mathrm{Re}(ir^3e^{i3\theta}+1)\\ &= \mathrm{Re}[ir^3(\cos(3\theta)+i\sin(3\theta))+1]\\ &= \mathrm{Re}(1-r^3\sin(3\theta)+ir^3\cos(3\theta))\\ &= 1-r^3\sin(3\theta).\end{aligned}$$

Since $-1 \le \sin(3\theta) \le 1$ for $0 \le \theta < 2\pi$ and $0 \le r < 2$, the supremum of $\mathrm{Re}(iz^3+1)$ is $1-2^3\times(-1) = 9$, because, e.g. the sequence of the real part of iz^3+1 where $z = re^{i\pi/2}$ goes to 9 as r goes to 2.)

1.3.4. (10 pts)

In general, if $\sin(z) = w$, then we have

$$\frac{e^{iz}-e^{-iz}}{2i} = w \quad \text{namely} \quad e^{iz}-e^{-iz} = 2iw.$$

Multiply the above equation by e^{iz}, we get the equation

$$X^2 - 2iwX - 1 = 0, \text{ where } X = e^{iz}.$$

So solve for X, we get

$$X = \frac{2iw \pm \sqrt{(2iw)^2+4}}{2} = iw \pm \sqrt{1-w^2},$$

where $\sqrt{1-w^2}$ means one of the square roots of $1-w^2$ (there are two of them who differ from each other by a minus sign).

(a) In this case, setting w to be $3/4+i/4$, by the above discussion, we have

$$e^{iz} = i\left(\frac{3}{4}+\frac{i}{4}\right) \pm \sqrt{1-\left(\frac{3}{4}+\frac{i}{4}\right)^2} = \left(\frac{3i}{4}-\frac{1}{4}\right) \pm \sqrt{\left(\frac{3}{4}-\frac{i}{4}\right)^2} = \left(\frac{3i}{4}-\frac{1}{4}\right) \pm \left(\frac{3}{4}-\frac{i}{4}\right).$$

So e^{iz} is either $1/2+i/2$ or $i-1$, that is, either $e^{i\pi/4}/\sqrt{2} = e^{i\pi/4-\ln\sqrt{2}}$ or $\sqrt{2}e^{i3\pi/4} = e^{i3\pi/4+\ln\sqrt{2}}$. This means that

$$iz = i\frac{\pi}{4} - \ln\sqrt{2} + 2k\pi i \quad \text{or} \quad iz = i\frac{3\pi}{4} + \ln\sqrt{2} + 2k\pi i. \qquad k \in \mathbb{Z}$$

Therefore, $z = \pi/4 + 2k\pi + i\ln\sqrt{2}$ or $z = 3\pi/4 + 2k\pi - i\ln\sqrt{2}$, $k \in \mathbb{Z}$.

(b) In this case, setting w to be 4, by the above discussion again, we have

$$e^{iz} = 4i \pm \sqrt{1-4^2} = 4i \pm \sqrt{15}i = (4 \pm \sqrt{15})e^{i\pi/2} = e^{i\pi/2+\ln(4\pm\sqrt{15})}.$$

So z is $\pi/2 + 2k\pi - i\ln(4 \pm \sqrt{15})$, $k \in \mathbb{Z}$.

1.3.10. (6 pts)

In order to find all the values of z such that $\sqrt{z^2} = z$, assume that $z = re^{i\theta}$, with $0 \le \theta < 2\pi$. Since the definition of '$\sqrt{\cdot}$' requires that the argument being used is within $[0, 2\pi)$, we argue in two cases.

(i) $0 \le \theta < \pi$.

In this case, $z^2 = r^2e^{i2\theta}$, where $0 \le 2\theta < 2\pi$. So by the definition, $\sqrt{z^2} = \sqrt{r^2}e^{i(2\theta)/2} = re^{i\theta} = z$ is always true.

(ii) $\pi \le \theta < 2\pi$.

In this case, $z^2 = r^2e^{i2\theta}$, where $2\pi \le 2\theta < 4\pi$. So by the definition, $\sqrt{z^2} = \sqrt{r^2}e^{i(2\theta-2\pi)/2} = re^{i\theta}e^{-\pi i} = -re^{i\theta} = -z$. For this to be true, $z = 0$.

Therefore, all the values of z satisfying $\sqrt{z^2} = z$ are of the form $z = re^{i\theta}$, where $r \ge 0$, and $0 \le \theta < \pi$. (i.e. the upper half plane with the non-negative part of the real axis).

1.3.14. (4 pts)

Because $e^{x+iy} = e^xe^{iy}$ with y being the argument of the complex number, when y is fixed and $x \to +\infty$, e^{x+iy} is going along the ray with argument y outwards towards ∞; when y is fixed and $x \to -\infty$, e^{x+iy} is going along the ray with argument y inwards towards the origin; when x is fixed and $y \to +\infty$, e^{x+iy} stays on the circle centered at the origin with radius e^x, winding about the origin counter-clockwise (with a period of 2π); when x is fixed and $y \to +\infty$, e^{x+iy} stays on the circle centered at the origin with radus e^x, winding about the origin clockwise (with a period of 2π).

1.3.22. (4 pts)

(a) The first quadrant is the collection of points $re^{i\theta}$ with $r > 0$ and $0 < \theta < \pi/2$. So the map $z \mapsto z^3$ take $re^{i\theta}$ to $r^3e^{i3\theta}$, with $0 < 3\theta < 3\pi/2$. So z^3 lies in the first three quadrants or the positive part of the y-axis or the negative part of the x-axis (name the union of these three D). Conversely, for any point in this D, one can write it as $re^{i\theta}$ with $0 < \theta < 3\pi/2$ and $r > 0$. And the map $z \mapsto z^3$ takes $\sqrt[3]{r}e^{i\theta/3}$, which is in the first quadrant, to this very point $re^{i\theta}$. Therefore, the image is the union of the first three quadrants and the positive part of the y-axis, and the negative part of the x-axis.

(b) For any interval of arguments of length 2π, for example $[\theta_0, \theta_0+2\pi)$, one can write any non-zero complex number uniquely as $re^{i\theta}$ with $\theta \in [\theta_0, \theta_0 + 2\pi)$. Under the map $z \mapsto \sqrt[3]{z}$, the modulus becomes the cubic root of the original modulus, and the argument is one-third of the original argument; thus the interval of arguments $[\theta_0, \theta_0 + 2\pi)$ would be mapped to $[\theta_0/3, \theta_0/3 + 2\pi/3)$. Therefore the image is a one-third sector (a union of two sextants including the boundary in the middle) and the starting boundary (starting from $\theta_0/3$).