SOLUTIONS FOR ASSIGNMENT 3 (MATH3X03)

2.1.2. (6pts)

(a) γ is the union of $\gamma_1(t) = it, t \in [0,1]$ and $\gamma_2(t) = i + 2t, t \in [0,1]$. So,

$$\begin{aligned}\int_\gamma x\,dz &= \int_{\gamma_1} x\,dz + \int_{\gamma_2} x\,dz \\ &= \int_0^1 0\gamma_1'(t)\,dt + \int_0^1 (2t)2\,dt \\ &= 2t^2\big|_0^1 = 2.\end{aligned}$$

(b) Since $F(z) = z^3/3 + z^2 + 3z$ is an entire function and $F'(z) = z^2 + 2z + 3$, we have

$$\int_\gamma (z^2 + 2z + 3)\,dz = F(2+i) - F(1) = \frac{29}{3} + \frac{32}{3}i - \frac{13}{3} = \frac{16}{3} + \frac{32}{3}i.$$

(c) One could have used the Deformation Theorem to reduce it to Example 2.1.12, except that the theorem is in §2.2. But the argument in Example 2.1.12 still works for this one. We can parameterize γ as $\gamma(\theta) = 1 + 2e^{i\theta}$ where $\theta \in [0, 2\pi]$. Then

$$\int_\gamma \frac{1}{z-1}\,dz = \int_0^{2\pi} \frac{1}{2e^{i\theta}} 2e^{i\theta} i\,d\theta = 2\pi i.$$

2.1.8. (5pts)

(a) One can parameterize γ as $\gamma(t) = (1+i)t$, where $t \in [0,1]$. So

$$\begin{aligned}\int_\gamma \bar{z}^2\,dz &= \int_0^1 \overline{(1+i)t}^2 (1+i)\,dt \\ &= (1-i)^2(1+i)\int_0^1 t^2\,dt = \frac{2}{3}(1-i).\end{aligned}$$

(b) γ can be parameterized as $\gamma_1(t) = t, t \in [0,1]$ joined with $\gamma_2(t) = 1 + it, t \in [0,1]$. So

$$\begin{aligned}\int_\gamma \bar{z}^2\,dz &= \int_0^1 t^2\,dt + \int_0^1 \overline{1+it}^2 i\,dt \\ &= \frac{1}{3} + i\left(t - \frac{t^3}{3} - it^2\right)_0^1 \\ &= \frac{4}{3} + \frac{2}{3}i.\end{aligned}$$

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In view of the answers to (a) and (b) and the Fundamental Theorem, $\bar{z}^2$ could not be the derivative of an analytic function (on an open set containing our two choices of γ), otherwise we should have got the same answer for (a) and (b).

2.1.10. (3pts)

It is easy to see that the length of C is $2\pi \times 2/4 = \pi$. So it is enough to show that on C, one always has

$$\left|\frac{1}{z^2+1}\right| \leq \frac{1}{3}.$$

But this is true by the Triangle Inequality:

$$\left|\frac{1}{z^2+1}\right| \leq \frac{1}{|z|^2-1} = \frac{1}{3}.$$

2.2.2. (3pts)

One way of seeing it is to use the anti-derivative of $1/z^2$. Since $F(z) = -1/z$ is analytic except at $z = 0$, by the Fundamental Theorem of Calculus for Contour Integrals, the integration $\int_\gamma 1/z^2\, dz$ is 0.

One can use a method similar to the one used in Example 2.2.8 using Deformation Theorem or using Example 2.2.9. But the argument essentially resorts to the Fundamental Theorm of Calculus.

2.2.6. (3pts)

γ is with in the domain of the principal branch of $\log(z)$. So if $F(z) = z^2/2 - \log(z)$, then $F'(z)$ is the integrand $z - 1/z$. So we conclude that

$$\int_\gamma (z - 1/z)\, dz = F(i) - F(1) = -1 - \log(i) = -1 - \frac{\pi}{2}i.$$

2.2.8. (3pts)

The polynomial $z^3(z^2+10)$ has 0 (a triple root) and $\pm i\sqrt{10}$ as its roots. They are all out side of the region enclosed by γ_1 and γ_2, which implies that the integrand is analytic on the region.[1] By the Deformation Theorem or (a method similar to the one used in) Example 2.2.9, one gets the desired equality.

2.3.8. (2pts)

This is essentially the same as Question 2.2.6.

$$\int_\gamma dz/z = \log(i) - \log(1) = \frac{\pi}{2}i.$$

[1] To be more precise, one really wants to pick a slightly bigger region for the analyticity of the integrand, containing the one enclosed by γ_1 and γ_2.