SOLUTIONS FOR ASSIGNMENT 4 (MATH3X03)

2.4.2. (3pts)

(a). By the equality

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right),$$

we have the following

$$\begin{aligned}\int_\gamma \frac{z^2-1}{z^2+1}\,dz &= \frac{1}{2i}\int_\gamma \frac{z^2-1}{z-i}\,dz - \frac{1}{2i}\int_\gamma \frac{z^2-1}{z+i}\,dz \\ &= \frac{1}{2i}2\pi i(i^2-1) - \frac{1}{2i}2\pi i((-i)^2-1) \quad \text{By Cauchy's Integral Formula} \\ &= 0.\end{aligned}$$

(b). The integration is $2\pi i \sin(e^0) = 2\pi i \sin(1)$.

2.4.4. (2pts)

For any z_0 inside γ, by Cauchy's Integral Formula,

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_\gamma 0\,dz = 0.$$

So $f = 0$ inside γ.

2.4.5. (2pts)

(a). Applying Cauchy's Integral Formula (for derivatives) to the function $f(z) = 1$ one gets 0 for the integral.

(b). The integral is $2\pi i/(3!)\sin^{(3)}(0) = -\pi i/3$.

2.4.6. (4pts)

Since f is analytic, f' is analytic too (see e.g. Page149). So

$$\begin{aligned}\int_\gamma \frac{f'(\zeta)}{\zeta - z_0}\,d\zeta &= f'(z_0)2\pi i I(\gamma, z_0) \quad \text{By Cauchy's Integral Formula} \\ &= f'(z_0)\frac{2\pi i}{1!}I(\gamma, z_0) \\ &= \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2}\,d\zeta. \quad \text{By CIF for derivatives}\end{aligned}$$

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Totally 25 points.

One way to generalize this is that, for all $k = 1, 2, \ldots$:

$$\int_\gamma \frac{f^{(k)}(\zeta)}{\zeta - z_0}\, d\zeta = f^{(k)}(z_0) 2\pi i I(\gamma, z_0)$$
$$= f^{(k)}(z_0)\frac{2\pi i}{k!} k! I(\gamma, z_0)$$
$$= k! \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta.$$

That is

$$\int_\gamma \frac{f^{(k)}(\zeta)}{\zeta - z_0}\, d\zeta = k! \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta, \quad k = 1, 2, \ldots.$$

2.4.8. (4pts)

The limit being 0 means that for every $\epsilon > 0$, there is some $N_\epsilon > 0$ such that whenever $|z| > N_\epsilon$, it is always true that $|f(z)/z| < \epsilon$. Since f is entire, we can use Cauchy's Inequalities (Theorem 2.4.7) for f'. For any z_0, let γ be the circle centered at z_0 with radius R bigger than $|z_0| + N_\epsilon$. It is then true that on γ, $|f(z)| < |z|\epsilon \leq 2R\epsilon$. So by Cauchy's Inequality for f', we have

$$|f'(z_0)| \leq \frac{1}{R} 2R\epsilon = 2\epsilon.$$

One can also see this by using (Proposition 2.1.6, without Theorem 2.4.7):

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2}\, dz \right| \leq \frac{1}{2\pi} \frac{2R\epsilon}{R^2} 2\pi R = 2\epsilon.$$

But this is true for any ϵ, as long as we enlarge the radius of γ accordingly; which must mean that $f'(z_0)$ is zero. So we conclude that f' is zero on $\mathbb{C}$; this in turn implies that f must be a constant on $\mathbb{C}$.

2.4.16. (2pts)

(a). This statement does not contradict Morera's Theorem. On the region $\mathbb{C}\backslash\{0\}$ (or any one contained in it), f satisfies both the condition and the conclusion of the theorem. On the region $\mathbb{C}$, f is not continuous; so it does not satisfy the condition of the theorem, and therefore it says nothing about the theorem.

(b). This statement does not contradict Liouville's theorem either, because $f(z)$ is not an entire function (nor is it bounded).

2.5.2. (3pts)

(One can of course use the strategy used in Example 2.5.15, here is another way.) The function $\cos(z)$ is clearly a non-constant entire function. So by the Maximum Modulus Principle (Theorem 2.5.6), the maximum of $|\cos(z)|$ is only attained on the boundary.

When $z = 0 + iy$, we have $|\cos(z)| = |e^{-y} + e^y|/2$, so the maximum $\cosh(2\pi)$ is attained when $y = 2\pi$.

When $z = 2\pi + iy$, we have $|\cos(z)| = |e^{-y} + e^y|/2$ again, so the maximum is still $\cosh(2\pi)$, when $y = 2\pi$.

When $z = x+i0$, we have $|\cos(z)| = |e^{ix}+e^{-ix}|/2 \leq (|e^{ix}|+|e^{-ix}|)/2 = 1$, so the maximum 1 is attained when $x = 0$.

When $z = x + i2\pi$, we have

$$|\cos(z)| = \frac{|e^{ix-2\pi} + e^{-ix+2\pi}|}{2} \leq \frac{|e^{ix-2\pi}| + |e^{-ix+2\pi}|}{2} = \frac{e^{-2\pi} + e^{2\pi}}{2} = \cosh(2\pi).$$

So the maximum $\cosh(2\pi)$ is attained when $x = 0$.

Finally, we conclude that the maximum for $|\cos(z)|$ on the given square is $\cosh(2\pi)$.

2.5.5. (2pts)

Consider the function $h(z) = f(z) - g(z)$. h is continuous on $\text{cl}(A)$ and analytic on A. So by the Maximum Modulus Theorem, $|h|$ has a finite maximum value on $\text{cl}(A)$ which is attained at some point on the boundary of A. But $|h|$ is actually 0 on the boundary of A. Since $|h| \geq 0$ for any point inside A (so the maximum is also attained at some point in A), h must be a constant on $\text{cl}(A)$, which is 0. That shows that $f = g$ on all of $\text{cl}(A)$.

2.5.7. (3pts)

It is clear that the function e^{z^2} is entire and non-constant. So the maximum modulus of it will be attained at some point on the unit circle. So if we write $z = e^{i\theta}, \theta \in [0, 2\pi]$ on the unit circle, then

$$|e^{z^2}| = |e^{\cos(2\theta)}e^{i\sin(2\theta)}| = |e^{\cos(2\theta)}| \leq e.$$

And $|e^{z^2}| = e$ if $z = \pm 1$.

So the maximum in question is e.