SOLUTIONS FOR ASSIGNMENT 5 (MATH3X03)

3.1.7. (4pts)

We remark that it is a well-known fact in classical analysis that for a series $\sum_{n=0}^{\infty} a_n$, if $\sum_{k=0}^{\infty} a_{2k}$ and $\sum_{k=0}^{\infty} a_{2k+1}$ converge to A_{even} and A_{odd} respectively, then the original $\sum a_n$ converges to $A_{\text{even}} + A_{\text{odd}}$. This fact applies to complex series as well. (And you should think about the proof.)

(a) Since $|i^n/\log n| > 1/n$, the series is not absolutely convergent. But both $\sum_{k=1}^{\infty} i^{2k}/\log(2k)$ and $\sum_{k=1}^{\infty} i^{2k+1}/\log(2k+1)$ converge by the alternating series test. Thus the series in question converges.

(b) The series does not absolutely converge but conditionally converges for the same reason.

3.1.10. (3pts)

If $\sum a_k$ converges, then for any $\epsilon > 0$, there exists some natural number N such that whenever $m, n > N$, we have

$$\left|\sum_{k=m}^{n} a_k\right| < \epsilon.$$

In particular, if $m = n$, then $|a_m| < \epsilon$ (for $m > N$). This is exactly the definition of $a_k \to 0, k \to \infty$.

If $\sum g_k(z)$ converges uniformly, then for any $\epsilon > 0$, there exists some natural number N such that whenever $m, n > N$, we have

$$\left|\sum_{k=m}^{n} g_k(z)\right| < \epsilon,$$

for all z (in the suitable domain). In particular, if $m = n$, then $|g_k(z)| < \epsilon$ for all z. This is exactly the definition of $g_k \to 0$ uniformly when $k \to \infty$.

3.1.14. (4pts)

First we note that for each n, the function $\frac{z^n}{1+z^{2n}}$ is analytic both on the interior and the exterior of the unit circle, since all the roots of $1 + z^{2n}$ are on the unit circle.

If $|z| \le \sigma$ for some σ such that $0 < \sigma < 1$. Then

$$\left|\frac{z^n}{1+z^{2n}}\right| \le \frac{\sigma^n}{|1+z^{2n}|} \le \frac{\sigma^n}{1-|z|^{2n}} \le \frac{\sigma^n}{1-\sigma^{2n}}.$$

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Totally 25 points.

By the root test, one knows that $\sum \sigma^n/(1-\sigma^{2n})$ converges if $0<\sigma<1$. Thus, by the Weierstrass M-Test (Theorem 3.1.7), the series in question is uniformly and absolutely convergent on the disc of radius σ centered at 0, which in turn, by the Analytic Convergence Theorem (Theorem 3.1.18), implies that it converges and represents an analytic function on the interior of the unit circle, since every closed disc contained in the interior of the unit circle is contained in the set $\{z \in \mathbb{C} \mid |z| < \sigma\}$ for some $0<\sigma<1$.

If $|z| \geq \delta$ for some $\delta > 1$. Then

$$\frac{z^n}{1+z^{2n}} = \frac{z^n/z^{2n}}{1/z^{2n}+1} = \frac{w^n}{1+w^{2n}},$$

where $w = 1/z$ and $|w| \leq 1/\delta$. Applying the above argument to w, we conclude[1] that the series in question also converges and represents an analytic function on the exterior of the unit circle.

3.2.2. (4pts)

(a) $R = \lim_{n\to\infty} n^2/(n+1)^2 = 1$.
(b) $R = \lim_{n\to\infty} \sqrt[2n]{4^n} = 2$.
(c) $R = \lim_{n\to\infty} (n!)/((n+1)!) = 0$.
(d) $R = \lim_{n\to\infty} \sqrt[n]{1+2^n} = 2$.

3.2.6. (3pts)

Since the function $1/(1+e^z)$ is analytic as long as $1+e^z \neq 0$, i.e. when $z \neq (2k+1)\pi i$, $k \in \mathbb{Z}$, the radius of convergence is $R = \pi$. (The nearest singularities to 0 is $\pm\pi$.)

To find the first four terms of the Taylor series around $z_0 = 0$, we do the following (there are many ways to do it, as usual). Suppose the Taylor series of it is

$$\frac{1}{1+e^z} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots .$$

By expanding e^z into its Taylor series, we have

$$1 = (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots)(1 + 1 + z + z^2/2 + z^3/6 + \cdots).$$

By comparing the coefficients on both sides, we get

$$2a_0 = 1,\ a_0 + 2a_1 = 0,\ a_0/2 + a_1 + 2a_2 = 0,\ a_0/6 + a_1/2 + a_2 + 2a_3 = 0.$$

Therefore,

$$\frac{1}{1+e^z} = \frac{1}{2} - \frac{1}{4}z + \frac{1}{48}z^3 + \cdots .$$

[1]Again, every closed disc contained in the exterior of the unit circle is contained in $\{w \in \mathbb{C} \mid 0 < |w| \leq 1/\delta\}$ for some $\delta > 1$; and then apply the Analytic Convergence Theorem. One thing you need to be careful about is how to handle 0 and ∞. But this argument just said takes care of it.

3.2.8. (4pts)

(a) $\sin z^2 = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z^2)^{2n-1}}{(2n-1)!} = \sum_{n=1}^{\infty}\frac{z^{4n-2}}{(2n-1)!}$, for all z.

(b) $e^{2z} = \sum_{n=0}^{\infty}\frac{2^n z^n}{n!}$, for all z.

3.2.18. (3pts)

Just follow the hint.

Since $\cos(0) \neq 0$, $\tan(z)$ is analytic at $z = 0$. So its Taylor expansion exists at $z = 0$ (in fact, with a radius of convergence $\pi/2$). Suppose the Taylor expansion is $a_0 + a_1 z + a_2 z^2 + \cdots$, then

$$\sin(z) = \cos(z)(a_0 + a_1 z + a_2 z^2 + \cdots).$$

That is

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \left(1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots\right)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 \cdots).$$

So we have

$$a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = \frac{1}{2}a_1 - \frac{1}{3!} = \frac{1}{3}.$$

That is

$$\tan(z) = z + \frac{1}{3}z^3 + \cdots .$$

3.2.20. (3pts Extra credit)

Suppose the series is $\sum a_n(z - z_0)^n$. On the circle of convergence, say with radius R, $\sum |a_n(z - z_0)^n| = \sum |a_n| R^n$ does not depend on the choice of z. Thus the series either absolutely converges everywhere or nowhere on its circle of convergence.

For example, $\sum z^n/n^2$ converges absolutely everywhere on its circle of convergence (of radius 1), while $\sum z^n/n$ does not absolutely converge anywhere on its circle of convergence (of radius 1).