SOLUTIONS FOR ASSIGNMENT 6 (MATH3X03)

3.3.2. (2 pts)

$$\begin{aligned}\frac{1}{z(z+1)} &= \frac{1}{z}\frac{1}{z+1} = \frac{1}{z}\frac{1}{z}\frac{1}{1+1/z} \\ &= \frac{1}{z^2}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n, \quad \text{when } |z|>1, \\ &= \sum_{n=0}^{\infty}(-1)^n z^{-n-2}.\end{aligned}$$

3.3.4. (4 pts)

We notice that

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right).$$

(a) When $0<|z|<1$, we have

$$\begin{aligned}\frac{1}{z(z-1)(z-2)} &= \frac{1}{z}\left(-\frac{1}{2}\frac{1}{1-z/2} + \frac{1}{1-z}\right) \\ &= \frac{1}{z}\left(-\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} + \sum_{n=0}^{\infty} z^n\right) \\ &= \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^{n-1}\end{aligned}$$

(b) When $1<|z|<2$, we have

$$\begin{aligned}\frac{1}{z(z-1)(z-2)} &= \frac{1}{z}\left(-\frac{1}{2}\frac{1}{1-z/2} - \frac{1}{z}\frac{1}{1-1/z}\right) \\ &= \frac{1}{z}\left(-\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}\right) \\ &= \sum_{n=0}^{\infty}\left(-\frac{1}{2^{n+1}}\right)z^{n-1} - \sum_{n=0}^{\infty}\frac{1}{z^{n+2}}.\end{aligned}$$

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Totally 32 points.

3.3.10. (3 pts)

Since f is analytic as stated and has a zero of order 1 only at z_0, there is some analytic $\phi(z)$ (on the same region) such that $f(z) = (z - z_0)\phi(z)$ and $\phi(z_0) \neq 0$. So then $f'(z) = \phi(z) + (z - z_0)\phi'(z)$. (We assume that z_0 is inside γ.)

Then

$$\begin{aligned}\int_\gamma \frac{zf'(z)}{f(z)}\,dz &= \int_\gamma \frac{z\phi(z) + z(z - z_0)\phi'(z)}{(z - z_0)\phi(z)}\,dz \\ &= \int_\gamma \frac{z}{z - z_0}\,dz + \int_\gamma \frac{z\phi'(z)}{\phi(z)}\,dz \\ &= \int_\gamma \frac{z}{z - z_0}\,dz, \quad \text{as the second integrand is analytic,} \\ &= 2\pi i z_0, \quad \text{by Cauchy's Integral Formula applied to the function } z.\end{aligned}$$

4.1.2. (8 pts)

(a) $z_0 = 1$ is not a zero of e^{z^2}, but it is a simple zero of $z - 1$, so it is a simple pole of the function in question. And by Proposition 4.1.2, we have

$$\text{Res}\left(\frac{e^{z^2}}{z - 1}, 1\right) = e^{z^2}|_{z=1} = e.$$

(b) 0 is not a singular point of the function, so the residue is 0.

(c) $z_0 = 0$ is a zero of $\cos(z) - 1$ of order 2, and a simple zero of z. Thus z_0 is a removable singularity. The residue is 0.

(d) z_0 is clearly a simple pole of this function. Thus

$$\text{Res}\left(\frac{z^2}{z^4 - 1}, e^{\pi i/2}\right) = \left.\frac{z^2}{4z^3}\right|_{z=e^{\pi i/2}} = \frac{1}{4}e^{-\pi i/2} = -\frac{i}{4}.$$

4.1.8. (5 pts)

(a) $e^z - 1 = 0$ if and only if $z = 2k\pi i$ for some $k \in \mathbb{Z}$. They are simple poles. So for all $k \in \mathbb{Z}$,

$$\text{Res}(1/(e^z - 1), 2k\pi i) = 1/e^{2k\pi i} = 1.$$

(b) $z = 0$ is the only place where the function is not defined; the function is analytic except at this point. Because when $z \neq 0$,

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} + \cdots,$$

we know that the residue at $z = 0$ is 1.

4.2.2. (2 pts)

$$\int_\gamma \frac{f(z)}{z - z_0}\,dz = 2\pi i\text{Res}\left(\frac{f(z)}{z - z_0}, z_0\right) I(\gamma, z_0) = 2\pi i f(z_0) I(\gamma, z_0),$$

with the last equality following from Proposition 4.1.2.

4.2.10. (8 pts)

(a) 0, since the integrand is analytic inside the circle.

(b) 0, since the integrand is analytic inside the circle.

(c) 0, by Cauchy's Integral Formula for Derivatives, or by the observation that the residue of the integrand is 0.

(d) $-\frac{2\pi i}{2!}e^1 = -e\pi i$, by Cauchy's Integral Formula for Derivatives.