Algebra 701, Autumn 2004, Homework Problems # 10

1) (Reid p. 68, 4.9) Let k be an arbitrary field, $A = k[X, Y, Z]/\langle X^2 - Y^3 - 1, XZ - 1\rangle$. Find $\alpha, \beta \in k$ such that A is integral over $B = k[X + \alpha Y + \beta Z]$. Write down a set of generators for A as a B-module.

2) (Reid p. 82, 5.2 (abridged)) Describe the irreducible components of $V(J) \subset k^3$ for each of the following ideals $J \subset k[X, Y, Z]$.

i) $\langle XY + YZ + XZ, XYZ\rangle$

ii) $\langle (X - Z)(X - Y)(X - 2Z), X^2 - Y^2 Z\rangle$

3) (Reid p.82, 5.6) Let k be an arbitrary field. For an ideal $J \subset k[X_1, \dots, X_n]$ and an extension field K of k, define a K-valued point of $V(J)$ to be a point $(a_1, \dots, a_n) \in K^n$ such that $f(a_1, \dots, a_n) = 0$ for all $f \in J$. State and prove a version of the Nullstellensatz in terms of K-valued points for all algebraic extension fields of k.

4) (Reid p. 82, 5.7) Let k be a field, K a Galois extension, G the Galois group of K over k. Prove that two K-valued points $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ of $V(J)$ correspond to the same maximal ideal of $k[X_1, \dots, X_n]$ if and only if there is $\sigma \in G$ such that $(a_1, \dots, a_n) = (\sigma(b_1), \dots, \sigma(b_n))$.