Algebra 701, Autumn 2004, Homework Problems # 9

1) (Reid p. 35, 1.15) Let k be a field, $a = (a_1, \dots, a_n) \in k^n$, and consider the evaluation map

$$e_a : k[x_1, \dots, x_n] \to k$$
$$e_a(f(x_1, \dots, x_n)) = f(a_1, \dots, a_n).$$

Prove that $\ker(e_a) = \langle x_1 - a_1, \dots, x_n - a_n \rangle$.
Hint: do this first for $a = (0, \dots, 0)$ and then apply a coordinate change.
Deduce that $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is a maximal ideal of $k[x_1, \dots, x_n]$.

2) (Reid p. 35, 1.16) Continue the above problem: suppose $a \in K^n$, where $k \subset K$ is an algebraic field extension. Determine the image and kernel of the evaluation map $e_a : k[x_1, \dots, x_n] \to K$. Let $I = \langle x_1 - a_1, \dots, x_n - a_n \rangle$ as an ideal in $K[x_1, \dots, x_n]$. Prove that $I \cap k[x_1, \dots, x_n]$ is a maximal ideal in $k[x_1, \dots, x_n]$.

3) (Reid p. 35, 1.17) Describe Spec$\mathbf{R}[X]$ in terms of $\mathbf{C}$. (List the irreducible polynomials in $\mathbf{R}[X]$ and describe how they factorise over $\mathbf{C}$.)

4) (Reid p.56, 3.11) Prove that if A is a noetherian ring then so is the formal power series ring $A[[X]]$. (Hint: as in the proof of the Hilbert basis theorem, consider the chain of ideals formed by the "leading coefficients". For a power series, its leading term is the the monomial of *lowest* degree.)