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GANZSTELLENSÄTZE IN THEORIES OF VALUED FIELDS

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Abstract

The purpose of this paper is to study an analogue of Hilbert's seventeenth problem for functions over a valued field which are integral definite on some definable set; that is, that map the given set into the valuation ring. We use model theory to exhibit a uniform method, on various theories of valued fields, for deriving an algebraic characterisation of such functions. We apply it to algebraically closed valued fields, model complete theories of difference and differential valued fields, and real closed valued fields. In the latter case, an essential step is to understand when a given valuation on $K(X)$ extending the valuation on the ordered valued field K is compatible with an ordering on $K(X)$. We characterise this completely in the case when X is a single variable.

1. *Introduction*

A "nichtnegativstellensatz" in real algebraic geometry is a theorem which gives an algebraic representation for a polynomial or rational function which takes only non-negative values on a given set. The original nichtnegativstellensatz is the solution by Artin [**1**] to Hilbert's seventeenth problem: a polynomial which takes only non-negative values everywhere can be written as a sum of squares of rational functions. The methods of Artin's solution are the foundations of the abstract study of real closed fields. Relativised versions of positivstellensätze and nichtnegativstellensätze describe the polynomials which are positive or non-negative on a semi-algebraic set in a real closed field. Abstract statements of these results can be found in the beautiful survey paper of Lam [**12**].

Model-theoretic proofs of the stellensätze rely on the fact that the theory of real closed fields is model complete. Dickmann [**6**] and Becker [**2**] use the model completeness of the theory of real closed rings to prove analogues of the stellensätze in this context (Dickmann for the functions which are globally non-negative, and Becker for some relativised statements). A real closed ring has a valuation, which gives rise to infinitesimal elements, and increases the collection of positive definite functions beyond just the sums of squares.

For valued fields in general one can consider the functional property of being integral instead of the property of being positive. That is, one asks for an algebraic representation of a function which always takes values in the ring of integers (all values of the function are "ganze Elemente"). Kochen [**10**] did exactly this for p-adically closed fields. The theorem of Kochen that we call a "ganzstellensatz" asserts that such an integral definite rational function (of course, no polynomial is integral definite) is integral over a ring defined using the Kochen γ operator. That is, the γ operator serves to characterise the integral elements in a p-adically

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closed field just as the squaring operator characterises the positive elements of a real closed field. More refined statements, and also a relative version, are given by Prestel and Roquette [**16**].

In this paper, we give a framework for a model-theoretic proof of such relative ganzstellensätze that can be followed in any model-complete theory of valued fields. We apply it to algebraically closed valued fields (Theorem 2.3), D-henselian valued D-fields (Theorem 3.1) and real closed valued fields (Theorems 4.12 and 4.13). Furthermore, we get similar representations for certain infinitesimal-definite functions (Corollary 5.2 and Proposition 5.3). An essential part of the general framework is a careful analysis in Subsection 2.2 of the concept of a function being integral. Bélair [**3**] has given a similar proof for the theory of Witt vectors. Indeed, it was he who pointed out to us the crucial lemma from Kochen's paper, and we thank him for thus making this work possible. We note that a similar flavour of model-theoretic nullstellensatz was given by Cherlin in [**4**].

A different motivation for these results, and the original reason that we started to ask these questions, comes from the problem of elimination of imaginaries. Given a definable set $S = \{x : v(p(x)) \geq v(q(x))\}$ in a model of a theory of valued fields, one would like to know if there is a canonical object (a "code") associated to S. Ideally, the code would be a tuple of elements from the field. The results of [**7**] and [**8**] show that, in the case of algebraically closed valued fields and p-adically closed fields, there is a code which is a sequence of definable modules and torsors. The results of the present paper show that, in each of the theories of valued fields that we consider, there is a ring associated to such a definable set S which contains all other functions which could be used to define the same set. Thus this ring is a canonical object associated to S. Of course, it is not itself definable, so does not solve the problem of elimination of imaginaries.

2. *The model theoretic framework*

We fix the following notation. For any valued field (K, v), let $\mathscr{O}_K$ be the valuation ring of K and $\mathscr{M}_K$ its maximal ideal. We will also often write $\mathscr{O}_v$ for the valuation ring of the valuation v and $\mathscr{O}_v^\times = \{x : v(x) = 0\}$ for its units. Extend v to K by letting $v(0) = \infty$. Let Γ denote the value group of v, and let $\Gamma_\infty = \Gamma \cup \{\infty\}$. Let k be the residue field, and denote the residue of an element x either by $\mathrm{res}(x)$ or by $\overline{x}$. Given an extension field L of K and an $\mathscr{O}_K$-subalgebra A of L, let $T = T(A) = \{1 + ma : a \in A, m \in \mathscr{M}_K\}$, and define $A_T = \{f \in L : tf \in A \text{ for some } t \in T\}$. Note that if $A \cap K = \mathscr{O}_K$ then T is a multiplicative set, and A_T is the localisation of A at T. However if $A \cap (K \setminus \mathscr{O}_K) \neq \emptyset$ then $A_T = L$ (since $0 \in T$). Let A_T^{int} be the integral closure of A_T in L.

The central piece of our framework is the following lemma (paraphrased) from Kochen.

LEMMA 2.1. *[**10**, Lemma 3] Let K be a valued field, L a field extension of K and A a subring of L such that $A \cap K = \mathscr{O}_K$. Then A_T^{int} is the intersection of all valuation rings $\mathscr{O}_L$ of L such that $A \subseteq \mathscr{O}_L$ and $\mathscr{O}_L \cap K = \mathscr{O}_K$.*

The proof in one direction is a straightforward calculation. For the other direction, one proves the contrapositive. Suppose that $x \in L$ is not in the integral closure of A_T. Consider the ideal $\mathscr{N}$ generated by $\mathscr{M}_K$ and x^{-1} in $A_T[x^{-1}]$. One can use the place extension theorem to show that there is a valuation $\tilde{v}$ on L whose maximal ideal contains $\mathscr{N}$ (and hence $\tilde{v}(x) < 0$). The assumption that $A \cap K = \mathscr{O}_K$ ensures that $\tilde{v}$ extends the given valuation on K.

2.1. *The naive framework*

The way that we want to use the above lemma is the following. Let $\mathcal{L}$ be a language which contains the language of valued fields. Assume for the moment that $\mathcal{L}$ does not add any function symbols. We will use $\mathbf{v}$ for the formal symbol in $\mathcal{L}$ interpreted as the valuation. However we might be less careful with the other symbols of $\mathcal{L}$, and employ the common practice of using them also for their various interpretations. Let $\mathcal{T}$ be a theory in the language $\mathcal{L}$, and assume it has a model completion $\widetilde{\mathcal{T}}$. In all our applications $\widetilde{\mathcal{T}}$ will actually admit quantifier elimination. Fix $\mathcal{K} = (K, v, \ldots)$ a model of $\widetilde{\mathcal{T}}$. So K is a valued field, possibly with extra structure. Let $S = \{x \in K^n : \mathcal{K} \models \varphi_S(x)\}$, where φ_S is a quantifier free formula in the language $\mathcal{L}$ with parameters from K. (See [**13**, Chapter 3] for definitions of all model theoretic notions, such as 'model completion' and 'quantifier elimination'.)

Definition 2.2. We say that a rational function $f(X) \in K(X) = K(X_1, \ldots, X_n)$ is *integral on* S if for all $x \in S$ we have $f(x) \in \mathscr{O}_K$.

We would like to use Lemma 2.1 to prove that there is an $\mathscr{O}_K$-subalgebra A of $K(X)$, defined in some way from S, such that for any $g(X) \in K(X)$,

$$g \text{ is integral on } S \text{ if and only if } g(X) \in A_T^{\text{int}} \, . \tag{1}$$

In order to do so, we need to construct the $\mathscr{O}_K$-algebra A to have the following properties.

Necessity $f \in A \Longrightarrow f$ is integral on S.

Sufficiency For any model $\mathcal{F} = \big(K(X), \tilde{v}, \ldots\big)$ of $\mathcal{T}$ which is an extension of $\mathcal{K}$, if $\tilde{v}(f(X)) \geq 0$ for all $f \in A$ then $\mathcal{F} \models \varphi_S(X)$.

Furthermore, A should contain a set $\mathscr{I}$, not dependent on S, with the *extension property*:

Extension For any valuation $\tilde{v}$ on $K(X)$ extending the valuation on K, if $\tilde{v}(f) \geq 0$ for all $f \in \mathscr{I}$ then $(K(X), \tilde{v})$ can be expanded to an $\mathcal{L}$-structure $\big(K(X), \tilde{v}, \ldots\big)$ which extends $\mathcal{K}$ and is a model of $\mathcal{T}$.

Given such an $\mathscr{O}_K$-algebra A, the standard argument using the model completeness of $\widetilde{\mathcal{T}}$ can now be applied to prove (1). If $A \cap (K \setminus \mathscr{O}_K) \neq \emptyset$ then $A_T = K(X)$, but we also get $S = \emptyset$ from necessity, so (1) holds. Hence we assume $A \cap K = \mathscr{O}_K$. The necessity condition gives the right-to-left direction, as localising at $T(A)$ and taking the integral closure preserve the property of being integral definite. To prove the forward implication, suppose that g is integral on S but $g(X) \notin A_T^{\text{int}}$. By Lemma 2.1, there is some valuation $\tilde{v}$ of $K(X)$ extending v such that $\tilde{v}(f) \geq 0$ for all $f \in A$ and $\tilde{v}(g) < 0$. Since $\mathscr{I} \subseteq A$ we can use the extension property to expand $\big(K(X), \tilde{v}\big)$ to a model $\mathcal{F} = \big(K(X), \tilde{v}, \ldots\big)$ of $\mathcal{T}$ which is an extension of $\mathcal{K}$. By the sufficiency property and as $\widetilde{\mathcal{T}}$ is the model completion of $\mathcal{T}$, there is $\mathcal{U}$ a model of $\widetilde{\mathcal{T}}$ which extends $\mathcal{F}$ and satisfies the following existential formula

$$\mathcal{U} \models \exists X \big(\varphi_S(X) \ \& \ \mathbf{v}(g(X)) < 0\big) \, .$$

By model completeness of $\widetilde{\mathcal{T}}$ the elementary submodel $\mathcal{K}$ satisfies the same formula, contradicting the hypothesis on g.

A paradigm for this argument is given by the case of algebraically closed valued fields. In this case, $\mathcal{L}$ is exactly the language of valued fields, $\mathcal{T}$ is the theory of valued fields (VF), and $\widetilde{\mathcal{T}}$ is the theory of algebraically closed valued fields (ACVF), which by Robinson [**18**] is the model completion of VF.

THEOREM 2.3. *Let $\mathcal{K} = (K, v)$ be an algebraically closed valued field. Let $S = \{x \in K^n : \bigwedge_{j\in J} f_j(x) \in \mathscr{O}_K\}$, where for each j in the finite enumeration set J, $f_j \in K(X)$. Let A be the $\mathscr{O}_K$-subalgebra of $K(X)$ generated by $\{f_j(X) : j \in J\}$. Then for any $g(X) \in K(X)$,*

$$g \text{ is integral on } S \iff g(X) \in A_T^{\text{int}} .$$

Proof. By the framework above, we need only verify that A satisfies the properties listed. The necessity is clear, as the process of taking the generated $\mathscr{O}_K$-algebra preserves non-negativity of the valuation. Also the sufficiency condition is clear, since the formula φ_S is just the integrality condition on the functions f_j which are in A. The set $\mathscr{I}$ of the extension property is empty in this case; any valuation on $K(X)$ which extends the valuation on K makes it a model of $\mathcal{T}$. □

REMARK 2.4. One can allow the index set J to be infinite if enough saturation is assumed. For example, if $\mathcal{K}$ is an $\aleph_1$-saturated model of ACVF then Theorem 2.3 holds with countable J (see [**13**, section 4.3] for the definition of $\aleph_1$-saturation).

REMARK 2.5. Prestel and Rippol [**15**] have related results with a similar proof. Their theorem is weaker, in that they only characterise functions which are integral definite on the valuation ring (that is, $S = \mathscr{O}_K$). It is stronger in two ways. First, they only assume that the field K is dense (in the valuation topology) in its algebraic closure, and second, they describe when the ring A_T is integrally closed. Theorem 2.3 could also be stated only assuming that K is dense in its algebraic closure; it just requires one extra step in the proof to use the continuity of the function g at a point in the algebraic closure witnessing the contradiction to find a nearby point in K which also witnesses the required contradiction. However, the analogous statement for D-henselian fields is less clear, so we prefer to preserve the uniformity of statement for the various theories.

2.2. *A refined concept of being integral*

The definition of what it means for a rational function to be integral at a point is somewhat sensitive. Kochen and Bélair both allow the rational function $f(X) = p(X)/q(X)$ to be integral at a point b where $f(X)$ is "not defined". It is clear that the function $\frac{1}{X-b}$ should not be considered to be integral at b, so presumably, this should be more precisely stated as allowing f to be integral at a point b satisfying $p(b) = q(b) = 0$, as this corresponds to b lying in the set $\{x : v(p(x)) \geq v(q(x))\}$. However it is not hard to show that, even with this definition, the set of rational functions integral at a point is not a ring, since we do not have cancellation in the semigroup Γ_∞. For example, if $c \notin \mathscr{O}_K$ then $\frac{X_1}{X_2} \cdot \frac{cX_2}{X_1}$ is not integral at $(0, 0)$.

On the other hand, there are contexts where we would like a function to be integral at a point where it is not defined. For example, in a valued D-field (see section 3) we would like the function $\frac{D(X-b)}{X-b}$ to be globally integral definite, so it should also be integral at b. Similarly, in an ordered valued field (see section 4) the function $\frac{X^2}{X^2+Y^2}$ should be integral everywhere, even at $(0,0)$.

The issue becomes more delicate with examples such as the following: should $f(X) = \frac{X-b}{D(X-b)}$ be integral at b? On the one hand there are points c arbitrarily close to b such that for $\varepsilon = c - b$ we have $D\varepsilon = \varepsilon$, whence $f(c) = \frac{\varepsilon}{\varepsilon} \in \mathscr{O}_K$. On the other hand there are c arbitrarily close to b such that $D(c-b) = 0$, whence $f(c) = \frac{\varepsilon}{0}$ is certainly not integral.

We therefore make the following definitions. Let $\mathcal{K} = (K, v, \ldots)$ be a model of $\widetilde{\mathcal{T}}$, and let F be either the field of rational functions $K(X)$ or, in the case of valued D-fields, the field of quotients of D-polynomials $K(X)_D$. Let $\tilde{v}$ be a valuation on F extending v. We will say that $\tilde{v}$ is a *$\mathcal{T}$-valuation* if there is an expansion of $(F, \tilde{v})$ to a model $\mathcal{F} = (F, \tilde{v}, \ldots)$ of $\mathcal{T}$ extending $\mathcal{K}$. In the case of valued D-fields, we require the expansion to interpret D on $F = K(X)_D$ in the standard way. Note that a VF-valuation is just a valuation on $K(X)$ extending v.

Definition 2.6. Let $b = (b_1, \ldots, b_n) \in K^n$. A $\mathcal{T}$-valuation $\tilde{v}$ on F is said to be *given by evaluation near b* if for any $1 \leq i \leq n$ and every $c \in K^\times$ we have $\tilde{v}(X_i - b_i) > v(c)$. We will also call such a $\tilde{v}$ a *$\mathcal{T}$-valuation near b* for short.

So each $X_i - b_i$ is a new infinitesimal smaller than any infinitesimal in $\mathscr{M}_K$. Note that in a valued D-field, as $\tilde{v}$ is assumed to be a VDF-valuation, $D(X_i - b_i)$ will also be a new small infinitesimal, as the theory VDF implies $\tilde{v}(D(X_i - b_i)) \geq \tilde{v}(X_i - b_i)$. An example where $\mathcal{T} = VF$ is the following valuation.

Let Γ be the value group of K and fix new elements $\delta_1, \ldots, \delta_n$ not in Γ. Let $\Gamma' = \Gamma \oplus \mathbb{Z}\delta_1 \oplus \ldots \oplus \mathbb{Z}\delta_n$ and make it an ordered group by setting $\delta_1 > \gamma$ for all $\gamma \in \Gamma$, and $\delta_{i+1} > \ell\delta_i$ for all ℓ in $\mathbb{N}$ $(i = 1, \ldots, n-1)$. We define a valuation $\tilde{v}$ of $K[X] = K[X_1, \ldots, X_n]$ into Γ' satisfying $\tilde{v}(X_i - b_i) = \delta_i$, which then extends canonically to $K(X)$. For any $p(X) \in K[X]$, write $p(X) = \sum_\alpha p_\alpha (X-b)^\alpha$ as a sum of monomials in $(X-b)^\alpha = \prod_{i=1}^n (X_i - b_i)^{\alpha_i}$ (here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index). Define

$$\tilde{v}(p(X)) = \min_\alpha \{v(p_\alpha) + \sum_{i=1}^n \alpha_i \delta_i\} \ .$$

Then $\tilde{v}$ is clearly a valuation on $K[X]$ (the argument from [**17**, Chapter 3A] applies, although the value group is different), and extends the valuation on K. In fact, it is clear that the above minimum is attained at the multi-index α which is minimal, with respect to the reverse lexicographic order, in the "support" $\{\alpha : p_\alpha \neq 0\}$ of p.

Notice that, if we let $b = (0,0)$ in the above example, then $\frac{cX_2}{X_1}$ is in $\mathscr{O}_{\tilde{v}}$, while $\frac{X_1}{X_2}$ is not. By the definition below, neither of these functions is VF-integral at $(0,0)$.

We now show that any (VF-) valuation given by evaluation near b is closely related to evaluation at b. A similar result will hold for $\mathcal{T}$-valuations near b for any theory $\mathcal{T}$ treated in the present paper.

Proposition 2.7. *Let $\tilde{v}$ be a valuation on $K(X)$ given by evaluation near b, and let $f \in K(X)$ be defined at b.*

(i) *If $f(b) \in K^\times$ then $\tilde{v}(f) = v(f(b))$.*

(ii) *If $f(b) = 0$ then $\tilde{v}(f) > 0$.*

Proof. Let $\delta_i = \tilde{v}(X_i - b_i)$ for $1 \leq i \leq n$. Let $f(X) = p(X)/q(X)$ for p and q coprime in $K[X]$ (so $q(b) \neq 0$), and write $p(X) = \sum_\alpha p_\alpha (X-b)^\alpha$. Note that for every $c \in K^\times$ and any $\alpha \neq (0, \dots, 0)$ we have $\tilde{v}(p_\alpha(X-b)^\alpha) = v(p_\alpha) + \sum_{i=1}^n \alpha_i \delta_i > v(c)$, hence $\tilde{v}(p(X) - p(b)) = \tilde{v}(p(X) - p_{(0,\dots,0)}) > v(c)$. If $p(b) \in K^\times$ then we get $\tilde{v}(p(X)) = v(p(b))$, and by the same result for $q(X)$ we get (i). If $p(b) = 0$ then $\tilde{v}(p(X)) > v(q(b)) = \tilde{v}(q(X))$, as required for (ii). □

Definition 2.8. Given f in F and $b \in K^n$, we say that the function f is $\mathcal{T}$-*integral at* b if, for every $\mathcal{T}$-valuation $\tilde{v}$ on F which is given by evaluation near b, we have $\tilde{v}(f) \geq 0$.

Remark 2.9.

(i) A function may be $\mathcal{T}$-integral at a point where it is not defined (and hence not integral). For example in the valued D-field context, given any D-polynomial p, the functions $\frac{Dp}{p}$ and $\frac{\sigma p}{p}$ are globally VDF-integral definite, even at roots of p. Similarly $\frac{X_1{}^2}{X_1{}^2+X_2{}^2}$ is OVF-integral everywhere, in particular also at $(0,0)$.

(ii) Note that $\frac{X-b}{D(X-b)}$ is not VDF-integral at b: there exists a VDF-valuation near b that gives $D(X-b)$ a strictly greater value than $X-b$.

We now change the above framework accordingly — we replace "integral" by "$\mathcal{T}$-integral" everywhere (specifically in (1) and in the necessity property). Note that, by "$\mathcal{T}$-integral on S" we simply mean "$\mathcal{T}$-integral at every point in S". We need to add the following assumption on $\mathcal{T}$-integrality.

Conservativity Whenever a function $f \in F$ is defined at a point $b \in K^n$, f is $\mathcal{T}$-integral at b if and only if f is integral at b.

There are three points that need restating. First, we note that $\mathcal{T}$-integrality is indeed preserved by the relevant operations (localising at $T(A)$ and taking integral closure), since each $\mathcal{O}_{\tilde{v}}$ is closed under these operations. Second, note that if the function g is given by $g(X) = p(X)/q(X)$, where p and q are coprime in $K[X]$ (or in $K[X]_D$), then actually $\mathcal{U}$ satisfies the formula

$$\mathcal{U} \models \exists X \left[\varphi_S(X) \ \& \ \big(\mathbf{v}(p(X)) < \mathbf{v}(q(X))\big) \ \& \ q(X) \neq 0 \right] .$$

Again by model completeness of $\widetilde{\mathcal{T}}$, $\mathcal{K}$ satisfies the same formula, and we get some $b \in S \subseteq K^n$ such that $g(b)$ is *defined* and non-integral. By the conservativity assumption we get that g is also not $\mathcal{T}$-integral at b, contradicting our revised hypothesis on g. Last, for the case $A \cap (K \setminus \mathcal{O}_K) \neq \emptyset$ we actually need existence of some $\mathcal{T}$-valuation near b (which follows from conservativity) in order to get a contradiction from the existence of some $b \in S$.

As an example we point out how to adapt Theorem 2.3 and its proof to the revised framework. In the statement of the theorem we replace "g is integral on S" by "g is VF-integral on S". We use Proposition 2.7 to show that VF-integrality is conservative. For the forward implication in conservativity we actually need the existence of some valuation near b, which we demonstrated by giving an example. In the proof of the revised theorem we note that, since each $f \in A$ is integral on S, it is also VF-integral there by conservativity, hence we get our revised necessity property.

2.3. *Integrality vs. VF-integrality*

We end this section by proving that when $\mathcal{T} = VF$ we do not get a new concept of integrality, so VF-integrality is actually equivalent to (naive) integrality. This makes the adaptation of Theorem 2.3 to the revised framework redundant. However the proof below that these concepts are equivalent is rather long, while the adaptation was straightforward, and will be easy to generalise.

Since VF-integrality is conservative all we need to show is that if $f \in K(X)$ is not defined at b then it cannot be VF-integral at b — there exists a valuation $\tilde{v}$ given by evaluation near b such that $f \notin \mathscr{O}_{\tilde{v}}$. Since one can easily shift to the origin we prove this for $b_0 = (0, \dots, 0) \in K^n$. For any valuation $\tilde{v}$ on $K(X)$ extending v denote the "relative valuation ring" by $R_{\tilde{v}/v} = \{f \in K(X) \ : \tilde{v}(f) \geq v(c) \text{ for some } c \in K\}$. We say that an element f is *$\tilde{v}$-large* if $f \notin R_{\tilde{v}/v}$, and that f is *$\tilde{v}$-small* if $1/f$ is $\tilde{v}$-large (or if $f = 0$). Note that $\tilde{v}$ is given by an evaluation near b_0 if and only if X_i is $\tilde{v}$-small for all $1 \leq i \leq n$, and that for such $\tilde{v}$ we have $K[X] \subseteq R_{\tilde{v}/v}$.

Proposition 2.10. *Let $p(X)/q(X) \in K(X) = K(X_1, \dots, X_n)$ be a rational function in reduced form such that $q(b_0) = 0$. Then there is a valuation $\tilde{v}$ given by evaluation near b_0 such that $p(X)/q(X)$ is $\tilde{v}$-large.*

Since $\mathscr{O}_{\tilde{v}} \subseteq R_{\tilde{v}/v}$ we get $p(X)/q(X) \notin \mathscr{O}_{\tilde{v}}$, so indeed the function p/q is not VF-integral at b_0, as stated above.

Proof. We prove the proposition by induction on n. We can reduce to the case that $q(X)$ is irreducible: let $\hat{q}$ be an irreducible factor of q such that $\hat{q}(b_0) = 0$, find a $\tilde{v}$ given by evaluation near b_0 such that $p/\hat{q}$ is $\tilde{v}$-large (so $p/\hat{q} \notin R_{\tilde{v}/v}$), and use $q/\hat{q} \in K[X] \subseteq R_{\tilde{v}/v}$.

Write $q(X) = Q(X_n) = \sum_{j=0}^{d} Q_j {X_n}^j$ where $Q_j \in K[X_1, \dots, X_{n-1}]$, assuming without loss $d > 0$. We intend to take a valuation w on $L = K(X_1, \dots, X_{n-1})$ given by evaluation near $(0, \dots, 0) \in K^{n-1}$, and extend it to a valuation $\tilde{v}$ on $L(X_n) = K(X)$ such that $\tilde{v}(Q(X_n)) > wL^\times$. This will make p/q $\tilde{v}$-large, as needed. The problem will be to do this in such a way that X_n is $\tilde{v}$-small. For example when $p(X)/q(X) = \frac{X_2}{X_2 X_3 - X_1}$ we will need X_1/X_2 to be w-small. We get the required control over w by the induction hypothesis.

Let $\alpha_1, \dots, \alpha_d$ be the roots of $Q(X_n) \in L[X_n]$ in the algebraic closure L^{alg} of L. To appreciate why we pass to the algebraic closure consider $\frac{X_1}{{X_2}^2 + X_1}$. We will find some valuation w on L^{alg} such that one of the roots α_ℓ is w-small, and such that the restriction of w to L is given by evaluation near $(0, \dots, 0) \in K^{n-1}$.

This is enough, as we can then extend w to the unique valuation $\tilde{v}$ on $L^{\text{alg}}(X_n)$ for which $\eta = \tilde{v}(X_n - \alpha_\ell) > w(L^{\text{alg}})^\times$ (so the value group of $\tilde{v}$ is $w(L^{\text{alg}})^\times \oplus \mathbb{Z}\eta$). Since $X_n - \alpha_\ell$ divides $Q(X_n) = q(X)$, and hence does not divide $p(X)$, we will get that p/q is $\tilde{v}$-large, as required. In addition, since $\tilde{v}(X_n - \alpha_\ell) = \eta > w(\alpha_\ell)$ (if $\alpha_\ell \neq 0$) we get $\tilde{v}(X_n) = w(\alpha_\ell)$, and since we assumed α_ℓ is w-small we get that X_n is $\tilde{v}$-small, as required (if $\alpha_\ell = 0$ then $\tilde{v}(X_n) = \eta > vK^\times$). Of course we actually need the restriction of $\tilde{v}$ to $K(X)$.

So assume for contradiction no such valuation w on L^{alg} exists. In other words, for any w such that $w|_L$ is given by evaluation near $(0, \dots, 0) \in K^{n-1}$ we have that none of the roots α_ℓ are w-small. In particular 0 is not a root, so $Q_0 \neq 0$. Rewriting

$Q(X_n) = \sum_{j=0}^{d} Q_j {X_n}^j$ in terms of $Z = 1/X_n$ we get

$$\sum_{j=0}^{d}(Q_j/Q_0)Z^{d-j} = \prod_{\ell=1}^{d}\left(Z - (1/\alpha_\ell)\right) .$$

Since all the inverses $1/\alpha_\ell$ are not w-large, so they are in the subring $R_{w/v}$ of L, we get that all the coefficients Q_j/Q_0 (for $1 \leq j \leq d$) are also in this subring. Note that $Q_0(0,\ldots,0) = q(b_0) = 0$, and let $\widehat{Q_0} \in K[X_1,\ldots,X_{n-1}]$ be any irreducible factor of Q_0 such that $\widehat{Q_0}(0,\ldots,0) = 0$. Since $Q_0/\widehat{Q_0} \in K[X_1,\ldots,X_{n-1}] \subseteq R_{w/v}$ we also get that all $Q_j/\widehat{Q_0}$ are in $R_{w/v}$.

Since any valuation on L extends to its algebraic closure we have shown that, for any valuation w on $L = K(X_1,\ldots,X_{n-1})$ given by evaluation near $(0,\ldots,0)$, $Q_j/\widehat{Q_0}$ is not w-large. By the induction hypothesis this has to mean that the irreducible $\widehat{Q_0}$ divides Q_j, hence $\widehat{Q_0}$ divides $Q(X_n) = q(X)$, but we assumed that q is irreducible. □

3. *D-Henselian valued fields*

In the formalism of Scanlon [**19**], a *valued D-field* is a valued field K with an additive function $D : K \to K$ satisfying the twisted Leibniz rule $D(xy) = xD(y) + yD(x) + eD(x)D(y)$, where e is a distinguished element of K with $v(e) > 0$. The function D and the valuation are required to have the following interaction: for all $x \in K$, $v(D(x)) \geq v(x)$. If we let $\sigma(x) = x + eD(x)$ then σ is a field endomorphism satisfying $v(\sigma(x)) = v(x)$ for all $x \in K$, and we can rewrite the above rule as $D(xy) = xD(y) + \sigma(y)D(x)$. If $e \neq 0$ then D and σ are interdefinable, and K is called a *valued difference field*. If $e = 0$ then D is a derivation, and K is called a *valued differential field* (in this case $\sigma = \mathrm{id}$). The notion of a D-henselian valued field is defined by Scanlon, and he proves (under certain hypotheses) [**19**, Theorem 7.1] that the theory of D-henselian valued fields has quantifier elimination and is the model completion of the theory VDF of valued D-fields. In this section, we take $\mathcal{L}$ to be the language of valued D-fields, and $\mathcal{T}$ a theory containing VDF which implies that the residue field is a differentially closed field of characteristic zero and the value group is divisible. Then Scanlon's results apply, and $\widetilde{\mathcal{T}}$ is the theory of D-henselian valued fields. The terms in the language can be written as elements of the ring of *D-polynomials* $K[\{D^k X_i\}_{1\leq i\leq n}^{k\geq 0}]$. Strictly speaking, we are not in the situation of the framework given above, as the functions of interest are quotients of D-polynomials, rather than rational functions. However, it is clear that the description of the framework can be modified to fit this situation.

Write $K[X]_D$ for the ring of D-polynomials in X and $K(X)_D$ for its field of fractions. We can think of $K[X]_D$ as polynomials in variables $\{Y_{i,k}\}_{1\leq i\leq n}^{k\geq 0}$ where $Y_{i,k}$ stands for $D^k X_i$. There is a unique extension of D to $K(X)_D$ such that $D(Y_{i,k}) = Y_{i,k+1}$ (when $e \neq 0$ this can be seen via σ).

Before stating our result about sets defined by weak valuation inequalities (Theorem 3.1) we take care of the general parts of our framework: we find a set $\mathscr{I}_D$ satisfying the extension property, and we show that the theory VDF satisfies the conservativity property. Let $\mathcal{K} = (K, v, D)$ be a valued D-field. Define

$$\mathscr{I}_D = \{Dp/p : p \in K[X]_D\} .$$

Using the formula $D(p/q) = (qDp - pDq)/q\sigma(q)$ (for $p, q \in K[X]_D$) it is easy to show that $\mathscr{I}_D$ has the extension property (the required expansion amounts to interpreting D on $K(X)_D$ in the standard way).

We now prove that VDF-integrality is conservative. Note that by Remark 2.9 (i) VDF-integrality is not equivalent to (naive) integrality. Let $\tilde{v}$ be a VDF-valuation given by evaluation near some $b \in K^n$, and let $f = p/q \in K(X)_D$. We can get a result similar to Proposition 2.7 by writing the D-polynomials $p\langle X\rangle$ and $q\langle X\rangle$ as polynomials in $\{D^k(X_i - b_i)\}_{1\le i\le n}^{k\ge 0}$ and using the fact that the theory VDF implies that $\tilde{v}(D^k(X_i - b_i)) \ge \tilde{v}(X_i - b_i) > vK^\times$. Hence in order to get conservativity it is enough to construct some VDF-valuation near b, and we may assume $b = b_0 = (0, \dots, 0)$.

Let $\Gamma' = \Gamma \oplus \mathbb{Z}\delta$ be ordered by setting $\delta > \gamma$ for all $\gamma \in \Gamma$. For a D-polynomial $p = \sum_\alpha p_\alpha Y^\alpha$ of order r in $Y^\alpha = \prod_{1\le i\le n}^{0\le k\le r} Y_{i,k}^{\alpha_{i,k}}$ define

$$\tilde{v}(p) = \min_\alpha \{v(p_\alpha) + |\alpha|\delta\}$$

(where $|\alpha| = \sum_{1\le i\le n}^{0\le k\le r} \alpha_{i,k}$), and extend $\tilde{v}$ to the fraction field $K(X)_D$. Then $\tilde{v}$ is clearly a valuation given by evaluation near b_0 (note that the restriction of $\tilde{v}$ to $K(X)$ does not yield the example given after Definition 2.6). We need to show $\tilde{v}$ is actually a VDF-valuation. It is easy to verify, using the twisted Leibniz rule, that the set $\{p \in K[X]_D : \tilde{v}(Dp) \ge \tilde{v}(p)\}$ is closed under multiplication, and it obviously contains $K \cup \{Y_{i,k}\}_{1\le i\le n}^{k\ge 0}$, since $\tilde{v}(Y_{i,k}) = \delta$. Hence the above set contains all monomials $p_\alpha Y^\alpha$, and we get

$$\tilde{v}(Dp) = \tilde{v}\big(\sum_\alpha D(p_\alpha Y^\alpha)\big) \ge \min_\alpha\{\tilde{v}(D(p_\alpha Y^\alpha))\} \ge \min_\alpha\{\tilde{v}(p_\alpha Y^\alpha)\} = \tilde{v}(p) \ .$$

The same condition for ratios p/q follows (as in the proof that $\mathscr{I}_D$ satisfies extension). We could have made $\langle \tilde{v}(Y_{i,k})\rangle_{k\ge 0}$ any non-decreasing sequence of values for each $1 \le i \le n$, as long as $\tilde{v}(Y_{i,0}) > \Gamma$.

Theorem 3.1. *Let $\mathcal{K} = (K, v, D)$ be a D-henselian valued field. Let $S = \{x \in K^n : \bigwedge_{j\in J} f_j\langle x\rangle \in \mathscr{O}_K\}$, where for each j in the finite enumeration set J, $f_j \in K(X)_D$. Let A be the $\mathscr{O}_K$-subalgebra of $K(X)_D$ generated by $\mathscr{I}_D \cup \{f_j\langle X\rangle : j \in J\}$. Then for any $g\langle X\rangle \in K(X)_D$,*

$$g \text{ is } VDF\text{-integral on } S \iff g\langle X\rangle \in A_T^{\mathrm{int}} \ .$$

Remark 3.2. Note that we do not replace integrality with VDF-integrality in the definition of the set S. The reason is that we need S to be (first-order) definable in order to use the model theory, and in the definition of $\mathcal{T}$-integrality we quantify over valuations. However by Remark 3.3 below we actually *can* redefine S in the manner suggested above, hence the model theory seems to allow us to go outside the first-order realm.

Proof. We need only verify that A satisfies the properties given in our revised framework (see subsection 2.2). For necessity, notice that each f_j is integral on S by definition, so by conservativity it is also VDF-integral on S. In addition all elements of $\mathscr{I}_D$ are globally VDF-integral, and VDF-integrality is preserved under the process of taking the generated $\mathscr{O}_K$-algebra. The sufficiency is as easy as in Theorem 2.3, however we give here more details. The formula $\varphi_S(x)$ says that each f_j is integral at x, so in particular defined there. If $f_j\langle X\rangle = p_j\langle X\rangle/q_j\langle X\rangle$ where p_j and q_j are relatively prime in $K[X]_D$ then $\varphi_S(x)$ is equivalent to

$$\bigwedge_{j\in J} \big(\mathbf{v}(p_j\langle x\rangle) \geq \mathbf{v}(q_j\langle x\rangle)\big) \ \&\ q_j\langle x\rangle \neq 0\ .$$

Now any valuation $\tilde{v}$ on $K(X)_D$ for which all elements of A are in $\mathscr{O}_{\tilde{v}}$ will in particular satisfy $\tilde{v}(f_j) \geq 0$, so for $\mathcal{F} = \big(K(X)_D, \tilde{v}, D\big)$ we have $\mathcal{F} \models \mathbf{v}(p_j\langle X\rangle) \geq \mathbf{v}(q_j\langle X\rangle)$. Since we also have $\mathcal{F} \models q_j\langle X\rangle \neq 0$ we get $\mathcal{F} \models \varphi_S(X)$, as required. We have already noted that $\mathscr{I}_D$ satisfies the extension property, and that VDF-integrality is conservative, so our revised framework applies. □

Remark 3.3.

(i) We get a natural inclusion-reversing correspondence associated to $\mathcal{T}$-integrality, analogous to null sets and radical ideals. For example in the valued D-field context, for any $A \subseteq K(X)_D$ let

$$V^{\text{int}}(A) = \{x \in K^n : \forall f \in A(f \text{ is } VDF\text{-integral at } x)\}$$

and for any $S \subseteq K^n$ let

$$I^{\text{int}}(S) = \{f \in K(X)_D : f \text{ is } VDF\text{-integral on } S\}\ .$$

We claim that the closure $V^{\text{int}}(I^{\text{int}}(S))$ of $S = \{x \in K^n : \bigwedge_{j\in J} f_j\langle x\rangle \in \mathscr{O}_K\}$ equals $\overline{S} = \{x \in K^n : \bigwedge_{j\in J} f_j \text{ is } VDF\text{-integral at } x\}$. First, since each f_j is integral on S by definition, it is also VDF-integral on S. Hence $\{f_j : j \in J\} \subseteq I^{\text{int}}(S)$, and then $\overline{S} = V^{\text{int}}(\{f_j : j \in J\}) \supseteq V^{\text{int}}(I^{\text{int}}(S))$. On the other hand, since each $Dp/p \in \mathscr{I}_D$ is globally VDF-integral (and the f_j are VDF-integral on $\overline{S}$ by definition) we get for A as in Theorem 3.1 that each $f \in A_{\mathcal{T}}^{\text{int}}$ is VDF-integral on $\overline{S}$. Hence $\overline{S}$ is contained in $V^{\text{int}}(A_{\mathcal{T}}^{\text{int}})$, which equals $V^{\text{int}}(I^{\text{int}}(S))$ by the above theorem. Since $A_{\mathcal{T}}^{\text{int}}$ is in the image of I^{int} we can conclude from $V^{\text{int}}(A_{\mathcal{T}}^{\text{int}}) = \overline{S}$ that $I^{\text{int}}(\overline{S}) = A_{\mathcal{T}}^{\text{int}}$, as stated in Remark 3.2.

(ii) The above arguments can be repeated for any theory $\mathcal{T}$ (satisfying the conservativity property) for which we have found a set $\mathscr{I}$ satisfying the extension property and such that every $f \in \mathscr{I}$ is globally $\mathcal{T}$-integral. Note that, although $\mathcal{T}$-integrality is non-elementary (at least a-priori — its given definition is not first-order), the resulting inclusion-reversing correspondence is quite reasonable, or "tame". For example compare $I^{\text{int}}(V^{\text{int}}(\{f_j : j \in J\})) = I^{\text{int}}(\overline{S}) = A_{\mathcal{T}}^{\text{int}}$ with the nullstellensatz.

4. *Real closed valued fields*

An *ordered valued field* is an ordered field $(K, <_K)$ with a convex subring. This convex subring is automatically the valuation ring of a valuation v on K which satisfies $0 <_K x <_K y \to v(x) \geq v(y)$. Ordered valued fields were studied by Cherlin and Dickmann [**5**], who proved that the model completion of the theory of ordered valued fields (OVF) is the theory of real closed valued fields (RCVF). The results of this section could be recast in the language of (real) valuation spectra of such fields (see [**9**]).

In our model-theoretic framework for the ganzstellensätze, the theory $\mathcal{T}$ is OVF and $\widetilde{\mathcal{T}}$ is RCVF. We aim to describe the collection of rational functions which are integral on definable sets S in a real closed valued field K. By quantifier elimination any definable set is a finite union of sets defined by conjunctions of formulae of the following forms: $v(f(x)) \geq 0$, $v(f(x)) > 0$, $p(x) \geq_K 0$, $p(x) >_K 0$ (where f is a rational function and p is a polynomial). We can assume our set is given by a conjunction, since the ring of functions integral on the union of several sets is given by the intersection of the appropriate rings.

In the present paper we consider the cases when S is defined either by a conjunction of formulae of the first type (which we refer to as the *pure* case) or by a single formula of the last type (the *mixed* case). In order to use our model-theoretic framework to prove ganzstellensätze in this context, we will need to find a set $\mathscr{I}$ satisfying the extension property. That is, we need to know in what circumstances a given valuation on $K(X)$ extending v is compatible with some linear order on $K(X)$ extending $<_K$. Our extension theorem (Theorem 4.1) is restricted to the field of rational functions in a single variable. Hence our ganzstellensätze only give algebraic characterisations for functions which are integral on definable subsets of the line. Furthermore, since a rational function in a single variable can be shown to be OVF-integral at $b \in K$ if and only if it is integral at b, we can revert in the rest of this section to the naive version of our framework, and ignore Definition 2.8. However note that for functions of more than one variable OVF-integrality is a new concept (see Remark 2.9 (i) for an example), even though it does satisfy the conservativity assumption.

In the literature, there is a treatment of the problems of characterising all valuations on a field compatible with a given ordering, and all orderings compatible with a given valuation (see in particular [**14**, section 7] and [**11**, Theorem 1.8 and section 3]). Our situation is more specific, in that we are concerned with the set of orderings which are compatible with a given valuation and extend the given ordering on the subfield K. Furthermore, in the mixed case (Theorem 4.13) we need to know the specifics of how these orderings are defined. However, we also have more flexibility, as we only need to know the existence of an appropriate ordering, rather than trying to characterise the family of such orderings.

4.1. *Expanding a valued field extension to an ordered valued field*

Let $\mathcal{K} = (K, v, <_K)$ be an ordered valued field. Denote the field of rational functions by $F = K(X)$, for X a single variable, and assume we are given a valuation v_F on F extending v with valuation ring $\mathscr{O}_F$. We want to examine the following question: can we extend the given ordering on K to an ordering on F in such a way that $\mathscr{O}_F$ is convex? In other words we ask whether (F, v_F) can be expanded to an ordered valued field $\mathcal{F} = (F, v_F, <_F)$ which extends $\mathcal{K}$. Suppose for the moment that such an expansion exists. Denote the set of positive elements in F by F_+, and note that the map $f \mapsto \frac{1}{1+f}$ takes F_+ into $(0,1) \subseteq \mathscr{O}_F$. Hence for all f in F, $\frac{1}{1+f^2} \in \mathscr{O}_F$. Of course we could replace f^2 here by any sum of squares. The aim of this subsection is to prove a converse statement when $\mathcal{K}$ is existentially closed, or equivalently a model of RCVF. We show that the above condition is the only obstruction to extending $<_K$ to F. In fact, we give a strong form of the converse: it is enough to know the above condition for linear polynomials. Thus the answer to the above question is given by the following theorem. Write Γ_F for the value group of the valued field F and k_F for its residue field.

Theorem 4.1. *Let $\mathcal{K} = (K, v, <_K)$ be a real closed valued field. Let $F = K(X)$ and assume that $\tilde{v}$ is a valuation on F extending v. Then $(F, \tilde{v})$ can be expanded to an ordered valued field $\mathcal{F} = (F, \tilde{v}, <_F)$ in such a way that $<_F$ extends $<_K$ if and only if*

$$\forall a, b \in K : \frac{1}{1 + (aX + b)^2} \in \mathscr{O}_{\tilde{v}} \ . \tag{2}$$

The proof of the theorem gives the following more detailed information: if $\Gamma_F \neq \Gamma$ then there are exactly two such ways to extend the ordering $<_K$ to F (and the condition is always true); if $k_F \neq k$ then there are (uncountably) many such ways to extend $<_K$ to F, unless F is an algebraic extension of K, in which case there is no appropriate extension of $<_K$; and if F is an *immediate extension* of K as a valued field (i.e. $\Gamma_F = \Gamma$ and $k_F = k$) then there is a unique such extension of $<_K$ to F.

Remark 4.2. By the remarks before the theorem, we only need to show that there is a suitable ordering on $K(X)$ if condition (2) is satisfied. Note that (2) has no content when $\tilde{v}(aX + b) \neq 0$.

It is useful to note the following simple lemma on an ordered valued field F.

Lemma 4.3.

(i) *If $v(\varepsilon) > v(a)$ then $sgn(a + \varepsilon) = sgn(a) \in \{+1, -1\}$.*

(ii) *If a and b have the same sign then they cannot cancel each other: if $sgn(a) = sgn(b)$ then $v(a + b) = min(v(a), v(b))$.*

Proof.

(i) Since $v(|\varepsilon|) = v(\varepsilon) > v(a) = v(|a|)$ we get $|\varepsilon| <_F |a|$, hence $a + \varepsilon$ has the same sign as a.

(ii) Assume for contradiction $v(a + b) > v(a)$. Since $v(-a) = v(a)$ we can use (i) to get $sgn(-a) = sgn((-a) + (a + b)) = sgn(b)$, contradicting our assumption.

□

The proof of the hard direction of Theorem 4.1 is given by the following sequence of lemmata. Lemmata 4.4 and 4.6 deal with the case where the value group has been extended, while Lemmata 4.7 through 4.11 deal with the complementary case $\Gamma_F = \Gamma$.

Lemma 4.4. *In the situation of Theorem 4.1, assume (2) holds. Suppose that $\Gamma_F \neq \Gamma$. Then there are $a, b \in K$ such that $\tilde{v}(aX + b) \notin \Gamma_\infty$.*

Proof. We assumed that $\tilde{v}\big(p(X)/q(X)\big) \notin \Gamma$ for some nonzero polynomials $p, q \in K[X]$. Since K is a real closed field we can write p and q as products of linear or irreducible quadratic factors. (Note that it is at this step that we have to assume we are working with polynomials in one variable.) If all these factors had 'old' values (i.e. values from Γ) then p/q would also have an old value; hence some have new values.

Assume for contradiction no linear term has a new value. Then $\tilde{v}\big(a(X+b)^2+c\big) \notin \Gamma$ for some $a, c \in K_+ = \{x \in K : x >_K 0\}$. Since we are assuming in particular that $X + b$ has an old value, the summands $a(X + b)^2$ and c both have old values. This has to mean that the new value is a result of cancellation, that is

$$\tilde{v}\big(a(X + b)^2\big) = \tilde{v}\big(c\big) < \tilde{v}\big(a(X + b)^2 + c\big) \ .$$

Hence $\tilde{v}\big(1 + [\sqrt{\tfrac{a}{c}}(X + b)]^2\big) > 0$, contradicting our assumption in (2). □

Now, since $K(aX + b) = K(X)$, we may assume that X itself has a new value $\tilde{v}(X) \notin \Gamma$.

Remark 4.5. The value group of a real closed valued field K is divisible — one can take p^{th} roots of any $x \in K$ when p is odd, and one can take square roots either of x or of $-x$, both of which have the same value.

Lemma 4.6. *In the situation of Theorem 4.1, assume (2) holds. Suppose that $\tilde{v}(X) \notin \Gamma$. Then there are exactly two ways to extend $<_K$ to an ordering $<_F$ on $F = K(X)$ such that $(F, \tilde{v}, <_F)$ is an ordered valued field.*

Proof. Assume we have chosen the sign of X, $sgn(X) = s \in \{+1, -1\}$. We show that this, together with the requirement that $K(X)$ be an ordered valued field, determines an ordering on $K(X)$.

Given a polynomial $p \in K[X]$, all its monomials have different values, since $\tilde{v}(X)$ is new and, by Remark 4.5, Γ is divisible. Hence exactly one of the monomials has least value, say $a_n X^n$, and we are forced by Lemma 4.3 (i) to let the sign of $p(X)$ be the sign of this monomial; $sgn(p(X)) := sgn(a_n)(sgn(X))^n = sgn(a_n)s^n$.

Let $\mathcal{P} = \{p(X) \in K[X] \ : \ sgn(p(X)) = +1\}$. We show that $\mathcal{P}$ is a positive cone; that is, it is closed under addition and multiplication. Let $p(X) = \sum_i a_i X^i$ and $q(X) = \sum_j b_j X^j$ be polynomials from $\mathcal{P}$, and assume that the monomials with least value in p and q are $a_n X^n$ and $b_m X^m$, respectively. Note that $\tilde{v}(p) = \tilde{v}(a_n X^n)$ and $\tilde{v}(q) = \tilde{v}(b_m X^m)$.

Closure under addition. First assume $\tilde{v}(p) = \tilde{v}(q)$. By divisibility of Γ we know that necessarily $n = m$. Since a_n and b_n have the same sign we cannot have cancellation (see Lemma 4.3 (ii)), so $v(a_n + b_n) = v(a_n)$, hence the monomial with least value in $p + q$ is $(a_n + b_n)X^n$. By definition we get $sgn(p + q) = sgn(a_n + b_n)s^n$. Now, since clearly $a_n + b_n$ shares the common sign of a_n and b_n, $p + q$ shares the common (positive) sign of p and q, as required.
Now assume without loss of generality that $\tilde{v}(p) < \tilde{v}(q)$. We get $\tilde{v}(a_n X^n) < \tilde{v}(b_m X^m) \leq \tilde{v}(b_n X^n)$, hence $v(a_n) < v(b_n)$, and thus $v(a_n + b_n) = v(a_n)$. It is easy to conclude that $(a_n + b_n)X^n$ has least value in $p + q$. By Lemma 4.3 (i) we get $sgn(p + q) = sgn(a_n + b_n)s^n = sgn(a_n)s^n = sgn(p) = +1$.

Closure under multiplication. Let $\ell = n + m$. The coefficient c_ℓ of X^ℓ in pq is the sum of $a_n b_m$ and 'smaller' terms, i.e. terms with higher value (by minimality). Hence c_ℓ will have the same value as $a_n b_m$. It follows that $\tilde{v}(c_\ell X^\ell) = \tilde{v}(a_n X^n) + \tilde{v}(b_m X^m)$, so $c_\ell X^\ell$ will be the monomial with least value in pq. In addition the sign of c_ℓ equals the sign of $a_n b_m$ (by Lemma 4.3 (i)). Hence we get

$$\begin{aligned} sgn(pq) = sgn(c_\ell X^\ell) = sgn(c_\ell)s^\ell &= sgn(a_n b_m)s^{n+m} \\ &= [sgn(a_n)s^n][sgn(b_m)s^m] \end{aligned}$$

$$= sgn(p)sgn(q) = +1,$$

as required.

Similar considerations now show that, if p and q are both positive and $\tilde{v}(p) < 0$, then $\tilde{v}(p+q) < 0$. Hence the valuation ring is convex in this ordering, as required. □

Now we wish to treat the case where there are no new values, so assume $\Gamma_F = \Gamma$. First we give a general lemma concerning the connection between 'good' orderings on a valued field extension L of K, and orderings on its residue field k_L.

Lemma 4.7. *Let $(K, v) \subseteq (L, \tilde{v})$ be any valued field extension such that $\Gamma_L = \Gamma$. Assume that K is an ordered valued field. Then there is a one-to-one correspondence between extensions of $<_K$ to L which make it into an ordered valued field, and extensions of the ordering on k to the residue field $k_L$of L.*

Proof. First the restriction of $<_L$ to the valued ring $\mathscr{O}_L$ of L respects the equivalence relation defined by $a \sim b$ iff $a - b \in \mathscr{M}_L$ (this is an easy consequence of Lemma 4.3 (i), for example). Hence any ordering on L making it an ordered valued field "descends" to an ordering on k_L.

Conversely, suppose we are given an ordering on k_L. Since L has no new values, any non-zero element $y \in L$ can be written as $y = \alpha z$, where $\alpha \in K$ has the same value as y, and hence $\tilde{v}(z) = 0$. Also we may assume α is positive, since $v(-\alpha) = v(\alpha)$. Now $res(z)$ is non-zero, so we can define $sgn(y) \overset{\text{def}}{=} sgn(res(z))$. This is well-defined, since if $\alpha z = \alpha' z'$ for positive α, α' then in k_L, $res(z) = res(\frac{\alpha'}{\alpha})res(z')$. Since $\frac{\alpha'}{\alpha}$ is positive and has valuation 0 its residue is also positive.

We need only to check that the set of positive y is closed under multiplication and under addition, so assume $\alpha, \beta \in K_+$, $\tilde{v}(z) = \tilde{v}(w) = 0$, and z, w have positive residues. We want to show that the sum $\alpha z + \beta w$ is positive (it is easily seen that the product $\alpha\beta(zw)$ is positive). Assume without loss of generality that $v(\alpha) \leq v(\beta)$, so $\frac{\beta}{\alpha} \in \mathscr{O}_K$. Write $\alpha z + \beta w = \alpha(z + \frac{\beta}{\alpha}w)$, and note that $res(z + \frac{\beta}{\alpha}w) = res(z) + res(\frac{\beta}{\alpha})res(w)$. Now $res(z)$ and $res(w)$ are positive, and since $\frac{\beta}{\alpha} \in K_+$ we know its residue is non-negative. Hence $res(z) + res(\frac{\beta}{\alpha})res(w)$ is positive, so in particular it's non-zero, therefore $z + \frac{\beta}{\alpha}w$ has valuation 0 and positive residue. Hence, by definition $\alpha(z + \frac{\beta}{\alpha}w)$ is indeed positive, as required. □

Corollary 4.8. *In the situation of Theorem 4.1, suppose that $F = K(X)$ is an immediate extension of K (so $\Gamma_F = \Gamma$ and $k_F = k$). Then there is a unique extension of $<_K$ to F making it an ordered valued field.*

We now give a lemma similar to Lemma 4.4, stating that a linear term has to be responsible for the extension, this time of the residue field.

Lemma 4.9. *In the situation of Theorem 4.1, assume (2) holds. Suppose k_F is a proper extension of the residue field of K. Then there is a linear term $aX + b \in \mathscr{O}_F^\times$ with a new residue.*

Proof. Assume for contradiction that for all $a, b \in K$, if $aX + b \in \mathscr{O}_F^\times$ then $res(aX + b) \in k$. We know that some $p(X)/q(X)$ in $\mathscr{O}_F^\times$ has a new residue. Write p

and q as products of linear factors and irreducible quadratic factors. Since $\Gamma_F = \Gamma$ we may rescale so that each of the factors is in $\mathcal{O}_F^\times$. Hence we know one of these factors has a new residue, and from our assumption we get that some irreducible quadratic Q has a new residue. Without loss of generality, we can write $Q = a(bX + c)^2 + d$ with $a, d \in K_+$ and $bX + c \in \mathcal{O}_F^\times$. Note that by our assumption $res(bX + c) \in k$.

If $d \in \mathcal{O}_K$ then a is in $\mathcal{O}_K$ too, and then $res(Q) = res(a)(res(bX + c))^2 + res(d)$ is old, contradicting Q's property. Hence $d \notin \mathcal{O}_K$, so we get $\tilde{v}\big([\sqrt{\frac{a}{d}}(bX+c)]^2+1\big) = \tilde{v}(Q) - \tilde{v}(d) = 0 - v(d) > 0$, contradicting (2). □

Since $K(bX + c) = K(X)$ we may assume that the linear term with a new residue is X itself.

Lemma 4.10. *In the situation of Theorem 4.1, assume (2) holds. Assume k_F is a proper extension of the residue field k of K and suppose $\overline{X} = res(X) \notin k$. Then k_F is the transcendental extension of k by $\overline{X}$.*

Proof. First we show that $k_F = k(\overline{X})$. The residue of a linear term $aX + b \in \mathcal{O}_F^\times$ is in $k(\overline{X})$, since if we assume (for contradiction) that $a \notin \mathcal{O}_K$ then we get $\tilde{v}(X + \frac{b}{a}) > 0$, hence $\overline{X} = -\mathrm{res}(b/a) \in k$. The residue of an irreducible quadratic is in $k(\overline{X})$ by the same argument used at the end of the proof of Lemma 4.9. Since K is a real closed field and there are no new values we can write any polynomial from $\mathcal{O}_F^\times$ as a product of linear and quadratic terms from $\mathcal{O}_F^\times$, and conclude that $k_F = k(\overline{X})$.

Now assume for contradiction that the extension is algebraic, i.e. $\overline{X}$ is algebraic over k. Since k is a real closed field this means $\overline{X}$ solves an irreducible quadratic. But then it is easy to get a contradiction to (2). □

Lemma 4.11. *In the situation of Theorem 4.1, assume (2) holds. Assume k_F is a proper extension of k and suppose $\overline{X} = res(X) \notin k$. Then we can add $\overline{X}$ into any Dedekind cut of $(k, <_k)$ that we wish, extend the ordering accordingly to $k_F = k(\overline{X})$, and "lift" the ordering from k_F to F making it an ordered valued field.*

Proof. Fix a Dedekind cut $\mathcal{D}$ in k (i.e. a partition of $(k, <)$ into two nonempty convex subsets). For any polynomial $p \in k[X]$, it is easy to find an interval (a, b) in k "bracing" the cut $\mathcal{D}$ (that is $[a, b]$ intersects both parts of the partition) such that p has constant sign on (a, b) (e.g. the endpoints may be chosen to be roots of p). Let $sgn(p)$ be this constant sign. The collection of "positive" polynomials is closed under addition and multiplication simply because the intersection of two open intervals bracing the cut $\mathcal{D}$ still braces the same cut. Now lift the resulting ordering from k_F to F using Lemma 4.7. □

Proof. (of Theorem 4.1)
To recapitulate, the previous lemmata give us the required result:

(i) If the value group is extended then a linear term $aX + b$ has a new value, and there are exactly two extensions of $<_K$ to $F = K(X)$ that give it the structure of an ordered valued field — we just need to choose the sign of $aX + b$.

(ii) If neither the value group nor residue field are extended then there is a unique appropriate extension of $<_K$ to F.
(iii) If the residue field is extended transcendentally then a linear term $aX+b$ has a new residue, and there are (uncountably) many appropriate extensions of $<_K$ to F — we can choose any Dedekind cut in k for $res(aX+b)$.
(iv) If the residue field is extended algebraically then condition (2) fails, and there is no appropriate extension of $<_K$ to F.

□

4.2. *The ganzstellensätze*

As mentioned in the preamble to this section, we prove ganzstellensätze for the two cases $S = \{x \in K : \bigwedge_{j \in J} v(f_j(x)) \geq 0\}$ and $S_p = \{x \in K : p(x) > 0\}$ where for each j in the finite set J, $f_j(X) \in K(X)$, and $p(X) \in K[X]$. For either case, let

$$\mathscr{I}_{ord} = \{\frac{1}{1+(aX+b)^2} : a, b \in K\} \ .$$

THEOREM 4.12. *Let $\mathcal{K} = (K, v, <_K)$ be a real closed valued field. Let $S = \{x \in K : \bigwedge_{j \in J} f_j(x) \in \mathscr{O}_K\}$. Let A be the $\mathscr{O}_K$-subalgebra of $K(X)$ generated by $\mathscr{I}_{ord} \cup \{f_j(X) : j \in J\}$. Then for any $g(X) \in K(X)$,*

$$g \text{ is integral on } S \iff g(X) \in A_T^{\text{int}} \ .$$

Proof. By the framework of section 2, we just need to show that A satisfies the given conditions. Sufficiency clearly follows from $f_j \in A$. Necessity is also clear, as any $f \in \mathscr{I}_{ord}$ is globally integral definite. Finally, the fact that $\mathscr{I}_{ord}$ satisfies the extension property is given exactly by Theorem 4.1. □

For the sets S that we have discussed up to this point, sufficiency was easily taken care of; it sufficed to ensure that the defining (D)-rational functions were included in A. In the mixed case, we get a slightly more complicated statement, and will actually not be able to find an $\mathscr{O}_K$-algebra A satisfying the sufficiency property. Instead we replace the sufficiency and extension properties by the following property of the $\mathscr{O}_K$-algebra A, which is readily verified to still fit our framework (see subsection 2.1).

Revised sufficiency Given a valuation $\tilde{v}$ on $K(X)$ extending v such that $\tilde{v}(f) \geq 0$ for all $f \in A$, we can expand $\big(K(X), \tilde{v}\big)$ to a model $\mathcal{F} = \big(K(X), \tilde{v}, \dots\big)$ of $\mathcal{T}$ such that $\mathcal{F} \models \varphi_S(X)$.

Let $S_p = \{x \in K : p(x) >_K 0\}$ for $p \in K[X]$. It is easy to see that, if $r(X) \in K[X]$ is any positive semidefinite function, then for all $s \in S_p$, $v\big(p(s)\big) \geq v\big(p(s)+r(s)\big) \neq \infty$, hence $\frac{p}{p+r}$ is integral on S_p. Define

$$\mathscr{I}_p = \{\frac{p}{p+r} : r(X) \in K[X] \text{ is a sum of squares}\} \ .$$

THEOREM 4.13. *Let $\mathcal{K} = (K, v, <_K)$ be a real closed valued field. Let $S_p = \{x \in K : p(x) >_K 0\}$, and let A be the $\mathscr{O}_K$-subalgebra of $K(X)$ generated by $\mathscr{I}_{ord} \cup \mathscr{I}_p$. Then for any $g(X) \in K(X)$,*

$$g \text{ is integral on } S_p \iff g(X) \in A_T^{\text{int}} \ .$$

Proof. As always, we need only verify that A satisfies the conditions of the framework. We noted that any function in $\mathscr{I}_p$ is integral on S_p, and necessity follows. The extension property for $\mathscr{I}_{ord}$ is Theorem 4.1. Thus it remains to verify the revised sufficiency condition.

Let $\tilde{v}$ be a valuation on $F = K(X)$ such that $f \in \mathscr{O}_{\tilde{v}}$ for all $f \in A$. We need to show that we can expand $(F, \tilde{v})$ to a model of OVF such that $0 <_F p(X)$. By Theorem 4.1 we know there is some ordering $<_F$ on F making it an ordered valued field; suppose $p = p(X) <_F 0$. We show that $<_F$ can be changed to make p positive. We rely heavily on the proof of Theorem 4.1.

Case 1: $\Gamma_F \neq \Gamma$; we may assume that $\tilde{v}(X)$ is new.

First suppose that the sign of p is determined by an odd monomial $a_{2n+1}X^{2n+1}$. Then by changing the sign of X (which we know we can) we are changing the sign of p, hence we can make p positive, as required.

Now assume the sign of p is determined by $a_{2n}X^{2n}$. Since the value of this monomial is strictly smaller than all the others, this means that $\tilde{v}(p - a_{2n}X^{2n}) > \tilde{v}(p)$. Since we assumed $p <_F 0$ we know $a_{2n} <_K 0$. Now we get $\tilde{v}(\frac{p}{p+(\sqrt{-a_{2n}}X^n)^2}) < 0$, contradicting $\mathscr{I}_p \subseteq A \subseteq \mathscr{O}_F$.

Case 2: $\Gamma_F = \Gamma$; in this case we may assume by rescaling p with $\alpha \in K_+$ that $\tilde{v}(p) = 0$.

First assume that $res(p) \in k$. Let $\alpha \in K$ have the same residue as p. Then since $p <_F 0$ we have $\alpha <_K 0$. But then we get $\tilde{v}(p + (\sqrt{-\alpha})^2) > 0 = \tilde{v}(p)$, contradicting $\mathscr{I}_p \subseteq \mathscr{O}_{\tilde{v}}$.

Now assume $res(p) \notin k$, and so in particular the residue field extends, and hence we know $res(aX+b) \notin k$ for some $a, b \in K$ such that $aX + b \in \mathscr{O}_F^\times$. We may assume that X itself has a new residue.

Claim All of p's coefficients are in $\mathscr{O}_K$, i.e. if $p(X) = \sum_{i=0}^d a_i X^i$ then for all i, $a_i \in \mathscr{O}_K$.

Proof of claim. If not, divide the above equality by a coefficient a_j having least value. Then since $\tilde{v}(p(X)/\alpha_j) = 0 - v(\alpha_j) > 0$ we get an equation for $res(X) = \overline{X}$ over k, $\sum_{i=0}^d res(\frac{a_i}{a_j})\overline{X}^i = 0$, contradicting the fact that $k \subseteq k(\overline{X})$ is a transcendental extension.

In order to make p positive in F it is enough to make $res(p)$ positive in k_F, and by the above claim $res(p) = \sum_{i=0}^d res(a_i)\overline{X}^i$. Assume for contradiction this is not possible. Since we know that we can put $\overline{X}$ into any cut of $(k, <)$, this means that $q(Y) = \sum_{i=0}^d res(a_i)Y^i$ is negative semidefinite. Since k is a real closed field, this can happen only if $-q$ is a sum of squares from $k(X)$. But then $-p$ is a sum of squares plus a polynomial with coefficients from the maximal ideal $\mathscr{M}_K$. Now, since $\tilde{v}(X) = 0$, this polynomial has positive value. Therefore we get $\tilde{v}(p + \text{sum of squares}) > 0 = \tilde{v}(p)$, contradicting $\mathscr{I}_p \subseteq \mathscr{O}_{\tilde{v}}$. □

Remark 4.14. Ideally, we would like to have a characterisation for functions which are integral definite relative to any definable set. Since we are in the case of one variable, the restriction in Theorem 4.13 to sets defined by a single polynomial inequality is not such a serious restriction. For any other order-definable set S can be written as $S_p \cup P$ for some set S_p as above and some finite set of points P. It is easy to show that $g \in K(X)$ is integral at a single point b if and only if $g(X) \in (\mathscr{I}_b + \mathscr{O}_K)_{\mathscr{I}_b+1}$, where $\mathscr{I}_b$ is the ideal generated by $X - b$ in $K[X]$.

For the special case $p = 1$, the result of Theorem 4.13 can be improved: by taking $S = K$ in Theorem 4.12, we see that the $\mathscr{O}_K$-algebra generated by $\mathscr{I}_{ord}$ is sufficient. One wonders whether the general statement of Theorem 4.13 could be improved similarly, by having r range only over squares of linear terms instead of sums of general squares. We show that such an improvement is impossible.

Let B be the $\mathscr{O}_K$-algebra generated by $\mathscr{I}_{ord} \cup \{\frac{p}{p+(aX+b)^2} : a, b \in K\}$. We give an example of a polynomial $p(X)$ and a function $g(X) \in K(X)$ such that g is integral on $S_p = \{x \in K : p(x) >_K 0\}$, but there is a valuation $\tilde{v}$ on $K(X)$ such that $\tilde{v}(g) < 0$ and $B \subseteq \mathscr{O}_{\tilde{v}}$, so $g \notin B_T^{\text{int}}$. Hence B can not replace A in Theorem 4.13.

Let $p = \varepsilon X^3 - X^2 - (X+1)^2$ for some non-zero $\varepsilon \in \mathscr{M}_K$. Then there are $a \in K$ s.t. $p(a) >_K 0$: take any a such that $v(a) < -v(\varepsilon)$ and $sgn(a) = sgn(\varepsilon)$ (e.g. $a = \varepsilon^{-3}$), and use Lemma 4.3 (i). Hence S_p is non-empty. Let $g(X) = \frac{p}{p+X^2+(X+1)^2}$. It is easy to see that g is integral on S_p (this is the easy direction of Theorem 4.13).

Now define $\tilde{v}$ on $K[X]$ by $\tilde{v}(\sum_i a_i X^i) = \min_i v(a_i)$ (so in particular $\tilde{v}(X) = 0$, and $res(X)$ is 'new', i.e. is not in k). Now, since $\tilde{v}(p) = 0$ and $\tilde{v}\big(p+X^2+(X+1)^2\big) = \tilde{v}(\varepsilon X^3) = v(\varepsilon) > 0$, we get $\tilde{v}(g) < 0$.

On the other hand we claim that $\tilde{v}\big(p + (aX+b)^2\big) \leq 0$ for any $a, b \in K$. Suppose not. Since $\tilde{v}(p) = 0$ we may assume that $\tilde{v}(aX+b) = 0$. Then by the definition of $\tilde{v}$ we get $a, b \in \mathscr{O}_K$. Thus in

$$p + (aX+b)^2 = \varepsilon X^3 + (a^2-2)X^2 + (2ab-2)X + (b^2-1)$$

we know that all the coefficients are from $\mathscr{O}_K$. We claim that at least one of these coefficients has valuation 0. If not then by taking the reduction to the residue field of K we get $\bar{a}^2 - 2 = 2\bar{a}\bar{b} - 2 = \bar{b}^2 - 1 = 0$, which is impossible.

Hence $\tilde{v}\big(p + (aX+b)^2\big) = 0$ by definition, and $\tilde{v}\big(\frac{p}{p+(aX+b)^2}\big) \geq 0$, as required. A similar (but much easier) argument shows that $\tilde{v}\big(\frac{1}{1+(aX+b)^2}\big) \geq 0$, hence $\tilde{v}$ is the required witness to $g \notin B_T^{\text{int}}$.

A similar argument works with X^4 instead of $X^2+(X+1)^2$, showing that squares of non-linear terms are also necessary in Theorem 4.13.

5. *Infinitesimal-definite functions*

We wish to give a similar treatment of (relative) *infinitesimal-definite functions*; that is, functions which map some definable set S into $\mathscr{M}_K$. A natural conjecture is that, for S defined by weak valuation inequalities, any such function equals some infinitesimal element times an integral-definite function. In this section we prove such a statement, in the ACVF and RCVF contexts, for functions of a single variable.

Proposition 5.1. *Let $\mathcal{K} = (K, v)$ be an algebraically closed valued field. Let $S = \{x \in K : \bigwedge_{j \in J} f_j(x) \in \mathscr{O}_K\}$, where $f_j \in K(X)$ for all j in some finite set J. Assume $g \in K(X)$ maps S into $\mathscr{M}_K$. Then there is some $\varepsilon > 0$ in Γ such that g maps S into $\varepsilon\mathscr{O}_K = \{x \in K : v(x) \geq \varepsilon\}$.*

Proof. Let R be the finite set of roots of numerators or denominators of one of the functions f_j or g.

We present S as a union $S = \bigcup_{r \in R} S_r$, where each set S_r satisfies the following three properties:

(i) The function $x \mapsto v(g(x))$ on S_r factors through $x \overset{\phi}{\mapsto} v(x-r) \overset{\psi}{\mapsto} v(g(x))$ (that is, $v(g(x))$ depends only on $v(x-r)$).
(ii) The map ψ above is piecewise $\mathbb{Z}$-linear, where the "pieces" are closed intervals.
(iii) The range of the map ϕ above, whose domain is S_r, is closed.

The existence of a suitable ε follows, since a $\mathbb{Z}$-linear image of a closed set is closed, and a closed subset of $\{v(x) : x \in \mathscr{M}_K\}$ is bounded away from 0.

Fix some $r \in R$. Define $U_r = \{x \in K : \forall r' \in R : v(x-r) \geq v(x-r')\}$, and note that $\bigcup_{r \in R} U_r = K$. Let $S_r = U_r \cap S$. Now define $\Delta_r = \{v(r'-r) : r' \in R\}$. Let $\Delta_r = \{\delta_1, \dots, \delta_{m_r}\}$ such that $\delta_1 < \delta_2 < \dots < \delta_{m_r}$ (note that $\delta_{m_r} = v(0) = \infty$). Let $\delta_0 =$"$-\infty$", and define (for $1 \leq i \leq m_r$) $V_r^i = \{x \in U_r : \delta_{i-1} \leq v(x-r) \leq \delta_i\}$. It is easy to verify that, for any $r' \in R$, $v(x-r')$ either equals $v(x-r)$ on V_r^i or is constant there. Thus on the set V_r^i we have that $v(g(x))$ is a $\mathbb{Z}$-linear function of $v(x-r)$, since $g(X) = c\prod_{r' \in R}(X-r')^{k_{r'}}$ for some $c \in K$ and $k_{r'} \in \mathbb{Z}$. Hence U_r satisfies properties (i) and (ii) above, and as a result its subset S_r satisfies (i) and (ii) as well.

Note that $\{v(x-r) : x \in U_r\} = \Gamma_\infty$. Since R also contains the zeros and poles of f_j we get that $v(f_j(x))$ is a piecewise $\mathbb{Z}$-linear function of $v(x-r)$ on U_r. Therefore intersecting U_r with $S = \{x \in K : \bigwedge_{j \in J} v(f_j(x)) \geq 0\}$ gives us $|J|$ closed conditions on $v(x-r)$, so the set $\{v(x-r) : x \in U_r \cap S\}$ is closed. Hence the set $S_r = U_r \cap S$ also satisfies property (iii) above, as required to complete the proof. □

We may choose any $e \in K$ with $v(e) = \varepsilon$, and note that $h(X) = g(X)/e$ is integral on S. Hence we can use Theorem 2.3 to get the following corollary.

Corollary 5.2. *With the notation of Theorem 2.3 for $n = 1$, g is infinitesimal-definite on S (that is, maps S into $\mathscr{M}_K$) if and only if $g(X) \in \mathscr{M}_K A_T^{\mathrm{int}}$.*

We now give an identical statement for RCVF.

Proposition 5.3. *Let $\mathcal{K} = (K, v, <_K)$ be a real closed valued field. Let $S = \{x \in K : \bigwedge_{j \in J} f_j(x) \in \mathscr{O}_K\}$, where $f_j \in K(X)$ for any j in some finite set J. Assume $g \in K(X)$ maps S into $\mathscr{M}_K$. Then there is some $\varepsilon > 0$ in Γ such that g maps S into $\varepsilon\mathscr{O}_K = \{x \in K : v(x) \geq \varepsilon\}$.*

Proof. The proof is very similar to that of Proposition 5.1, so we only indicate the necessary adjustments. First we explain how to modify R. Here every function f_j, as well as g, can be written as a product of linear and irreducible quadratic terms and their inverses. For any quadratic term $Q_i = (X-c_i)^2 + d_i^2$ appearing in one of the relevant functions we add c_i to the set R. We also add $v(d_i)$ to each Δ_r for $r \in R$. We then proceed in exactly the same way — on each set V_r^i we conclude that $v(f_j(x))$ is a linear function of $v(x-r)$, and similarly for $v(g(x))$. □

Remark 5.4. An alternative proof of the above proposition is to reduce to the easy special case $S = K$, by applying Lemma 4.3 (ii) to the squares of the functions f_j and of $1/g$.

Again we get as an easy corollary, this time of Theorem 4.12, that $g \in K(X)$ is infinitesimal-definite on S iff $g(X) \in \mathscr{M}_K A_T^{\text{int}}$, where this time A is the $\mathscr{O}_K$-algebra generated by $\mathscr{I}_{ord} \cup \{f_j(X) : j \in J\}$.

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