One-dimensional p-adic subanalytic sets

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1 Introduction

In this article we extend two theorems from [2] on p-adic subanalytic sets, where p is a fixed prime number, $\mathbf{Q}_p$ is the field of p-adic numbers and $\mathbf{Z}_p$ is the ring of p-adic integers. One of these theorems, 3.32 in [2], says that each subanalytic subset of $\mathbf{Z}_p$ is semialgebraic. This is extended here as follows.

Theorem A *Let $S \subseteq \mathbf{Z}_p^{m+1}$ be a subanalytic set. Then there is a semialgebraic set $S' \subseteq \mathbf{Z}_p^{m'+1}$ such that for each $x \in \mathbf{Z}_p^m$ there is $x' \in \mathbf{Z}_p^{m'}$ with $S_x = S'_{x'}$.*

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Hence the semialgebraic complexity of the fibers $S_x = \{y \in \mathbf{Z}_p : (x, y) \in S\}$ remains bounded as the parameter x ranges over $\mathbf{Z}_p^m$.

The proof of 3.32 in [2] depends on the compactness of $\mathbf{Z}_p$ in a way that makes it hard to see how it could be adapted to give the stronger Theorem A. Instead we take a model-theoretic approach and work in nonstandard models to prove the P-minimality of the analytic theory of $\mathbf{Q}_p$ in the sense of [5], which implies Theorem A. To explain this we now introduce some notation used throughout the paper.

Let K be a field complete with respect to a non-trivial nonarchimedean absolute value $|\,|$, for instance $K = \mathbf{Q}_p$; let $R = \{x \in K : |x| \leq 1\}$ denote its valuation ring. Let $Y = (Y_1, ..., Y_n)$ be a tuple of distinct indeterminates, and let $K\langle Y\rangle$ be the ring of power series $f(Y) = \sum_\nu a_\nu Y^\nu \in K[[Y]]$ such that $|a_\nu| \to 0$ as $|\nu| \to \infty$ (the Tate ring in Y over K). Here $\nu = (\nu_1, \ldots, \nu_n)$ ranges over multi-indices in $\mathbf{N}^n$ and $|\nu| = \nu_1 + \ldots + \nu_n$. Given such a power series $f(Y)$ we define a function $f\ : K^n \to K$ by

$$f(y) = \begin{cases} \sum_\nu a_\nu y^\nu, & \text{for } y \in R^n \\ 0, & \text{otherwise.} \end{cases}$$

We call such a function a *restricted analytic function on* K^n. For technical reasons (closure under substitution) we will also work with the subring

$$R\langle Y\rangle := K\langle Y\rangle \cap R[[Y]]$$

of $K\langle Y\rangle$. The restricted analytic function on K^n given by a power series $f(Y) \in R\langle Y\rangle$ maps R^n into R. Let the language $\mathcal{L}_R$ be obtained from the language of rings $\{0, 1, +, -, \cdot\}$ by adding a unary relation symbol, and for each $f \in R\langle Y_1, \ldots, Y_n\rangle$ with $n \in \mathbf{N} = \{0, 1, 2, \ldots\}$ a new n-ary function symbol, which for simplicity we also denote by f. We consider K as an $\mathcal{L}_R$-structure by interpreting $0, 1, +, -, \cdot$ as the usual ring operations, the unary relation symbol as the valuation ring, and each new function symbol f as the corresponding restricted analytic function; this makes R an $\mathcal{L}_R$-substructure of K.

Suppose now that $K = \mathbf{Q}_p$, so $R = \mathbf{Z}_p$; in this situation we add to the language $\mathcal{L}_{\mathbf{Z}_p}$ unary predicate symbols P_r for $r > 0$ to obtain the language $\mathcal{L}_{an}$; we interpret them in $\mathbf{Q}_p$ and $\mathbf{Z}_p$ by

$$P_r(x) \text{ if and only if } x = y^r \text{ for some nonzero } y.$$

Finally we extend $\mathcal{L}_{an}$ to $\mathcal{L}_{an}^D$ by adding a new binary function symbol D to be interpreted in $\mathbf{Q}_p$ and $\mathbf{Z}_p$ as 'restricted division':

$$D(x, y) = \begin{cases} \frac{x}{y}, & \text{if } 0 < |x| \leq |y| \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

These two extensions of the language $\mathcal{L}_{\mathbf{Z}_p}$ do not enlarge the class of definable relations but they are convenient since $\mathbf{Q}_p$ and $\mathbf{Z}_p$ admit elimination of quantifiers in the language $\mathcal{L}_{an}^D$, and the subanalytic subsets of $\mathbf{Z}_p^n$ are exactly the subsets of $\mathbf{Z}_p^n$ that are (quantifier-free) definable in the language $\mathcal{L}_{an}^D$. (See [2].) We can now state the model-theoretic equivalent to Theorem A, the so-called P-minimality of the $\mathcal{L}_{an}$-theory of $\mathbf{Q}_p$.

Theorem A′ *Let K_1 be an $\mathcal{L}_{an}$-elementary extension of $K = \mathbf{Q}_p$. Then each definable subset of K_1 is semialgebraic.*

Here and throughout 'definable' means 'definable with parameters'. By 'semialgebraic' we mean 'quantifier-free definable in the language of rings augmented by the power predicates P_n'; Macintyre's quantifier elimination theorem [7] implies that 'semialgebraic' coincides with 'definable in the language of rings'.

It is clear that Theorem A implies Theorem A′; the converse follows by a model-theoretic compactness argument. To prove Theorem A′ (and thereby Theorem A) we introduce in section 3 the ring $\mathcal{O}(F)$ of 'holomorphic' functions on a 'connected affinoid' F as a way of understanding 1-variable terms in $\mathcal{L}_{an}^D$ with nonstandard parameters as piecewise holomorphic functions. We establish properties of $\mathcal{O}(F)$ similar to those known in the standard theory developed in [4, I.2 and II.4]. (In our situation the connected affinoids live in a nonstandard model, and some arguments of [4] have to be modified.) In section 4 we use these properties to show for a given 1-variable term $t(Z)$ with parameters from the valuation ring R_1 of K_1 that there are connected affinoids $F_1, \ldots, F_k$ covering R_1 and functions $f_1 \in \mathcal{O}(F_1), \ldots, f_k \in \mathcal{O}(F_k)$ such that $t(z) = f_i(z)$ for all but finitely many $z \in F_i \cap R_1$, for $i = 1, \ldots, k$. This goes by induction on the complexity of the term $t(Z)$, the trouble being caused by occurrences of the symbol D. Theorem A′ then follows easily.

In section 5 we prove an analogue for arbitrary models of the result from [2] that 1-dimensional subanalytic subsets of $\mathbf{Z}_p^n$ are semianalytic. We establish a characterization of 1-variable definable functions (in arbitrary models) in terms of nth roots of holomorphic functions on connected affinoids. This result seems to be new even for the standard model. We refer to section 5 for detailed statements.

Lipshitz and Robinson [6] recently established the analogue of Theorem A for rigid subanalytic sets. Their method is quite different, as they are able to use previous work on 'holomorphic functions' on 'R-domains' to show directly that there is a bound on the complexity of fibers, independent of the parameters.

2 Weierstrass preparation in elementary extensions of the p-adic field

From now on we assume that our complete nonarchimedean field K has value group $vK = \mathbf{Z}$, where the valuation v on K has valuation ring $R = \{x \in K : |x| \leq 1\}$. We also fix an element $t \in R$ with $v(t) = 1$, so that tR is the maximal ideal of R, and we let $\bar{K} = R/tR$ denote the residue field. Only in sections 4 and 5 do we specialize to the case $K = \mathbf{Q}_p$.

Our purpose in this section is to define, for an elementary extension K_1 of K, a ring of functions $K_1\{Y\}$ analogous to the Tate ring, and to prove a Weierstrass Preparation for it. Since functions in this ring can be obtained by instantiating parameters, we need a parametric version of Weierstrass Preparation in the standard model (which is first order so transfers to K_1). The parametric version is implicit in [2] and explicit in [3], but for the reader's convenience we shall quote in 2.1 below some facts from [3].

We fix some notation for use throughout the paper. We will always let $Y = (Y_1, \ldots, Y_n)$, and put $Y' := (Y_1, \ldots, Y_{n-1})$ for $n > 0$. We also use $X = (X_1, \ldots, X_m)$ and occasionally U and V for tuples of distinct (parametric) variables. Typically, we write $f(X,Y) \in K\langle X, Y\rangle$ as $f(X,Y) = \sum a_\nu(X)Y^\nu$ with $a_\nu(X) \in K\langle X\rangle$ for all $\nu \in \mathbf{N}^n$, and we substitute elements $x \in R^m$ for X to get a power series $f(x,Y) = \sum a_\nu(x)Y^\nu \in K\langle Y\rangle$. The index set $\mathbf{N}^n$ is lexicographically ordered. Each $f(Y) = \sum a_\nu Y^\nu \in R\langle Y\rangle$ gives rise to a polynomial $\bar{f}(Y) = \sum \bar{a}_\nu Y^\nu \in \bar{K}[Y]$ over the residue field. If $n > 0$ we say that such an f is *regular in* Y_n *of degree* q if $\bar{f}$ has the form

$$cY_n^q + \sum_{i<q} c_i Y_n^i,$$

where $c \in \bar{K}$, $c \neq 0$, and $c_0, \ldots, c_{q-1} \in \bar{K}[Y']$.

2.1 Parametric Weierstrass Preparation

Fix

$$f(X,Y) = \sum_{\nu \in \mathbf{N}^n} a_\nu(X)Y^\nu \in R\langle X, Y\rangle,$$

with $n > 0$. By Lemma 1.5 of [3], there is $d > 0$ such that for all $\nu \in \mathbf{N}^n$ with $|\nu| \geq d$ we have $a_\nu = \sum_{|\mu|<d} b_{\nu\mu}a_\mu$, where $b_{\nu\mu} \in tR\langle X\rangle$ can be chosen such that $||b_{\nu\mu}||$, the supremum of the coefficients, converges to 0 as $|\nu| \to \infty$, for each fixed μ with $|\mu| < d$. For each $\mu \in \mathbf{N}^n$ with $|\mu| < d$, define the finite sets

$$A(\mu) := \{\nu \in \mathbf{N}^n : |\nu| < d, \nu < \mu\},$$
$$B(\mu) := \{\nu \in \mathbf{N}^n : |\nu| < d, \nu > \mu\},$$

and introduce tuples of indeterminates $U_\mu := \big(U_{\mu\nu} : \nu \in A(\mu)\big)$ and $V_\mu := \big(V_{\mu\nu} : \nu \in B(\mu)\big)$. Also put

$$f_\mu := Y^\mu + \sum_{|\lambda| \geq d} b_{\lambda\mu} Y^\lambda.$$

Then $f_\mu \in R\langle X, Y\rangle$, and

$$f = \sum_{\nu \in A(\mu)} a_\nu f_\nu + a_\mu f_\mu + \sum_{\nu \in B(\mu)} a_\nu f_\nu.$$

Define

$$\tilde{F}_\mu := \sum_{\nu \in A(\mu)} U_{\mu\nu} f_\nu + f_\mu + \sum_{\nu \in B(\mu)} t V_{\mu\nu} f_\nu,$$

let $T_d(Y) := (Y_1 + Y_n^{d^{n-1}}, \dots, Y_{n-1} + Y_n^d, Y_n)$ and put

$$F_\mu := \tilde{F}_\mu(U_\mu, V_\mu, X, T_d(Y)) \in R\langle U_\mu, V_\mu, X, Y\rangle.$$

Then F_μ is regular in Y_n of degree $\ell := \mu_1 d^{n-1} + \ldots + \mu_n$.

We define $\mathcal{L}_R$-formulas $Z(X)$ and $S_\mu(X)$ (for $|\mu| < d$) by:

$$Z(X) := \bigwedge_{|\mu|<d} a_\mu(X) = 0,$$

$$S_\mu(X) := a_\mu(X) \neq 0 \wedge \bigwedge_{\nu \in A(\mu)} |a_\nu(X)| \leq |a_\mu(X)| \wedge \bigwedge_{\nu \in B(\mu)} |a_\nu(X)| < |a_\mu(X)|.$$

The following (implicit in [2]) is Remark 1.13 of [3].

Lemma 2.1 **(i)** $K \models \forall x \in R^m \big(Z(x) \vee \bigvee_{|\mu|<d} S_\mu(x) \big)$.

(ii) *There are $c_1, \dots, c_\ell \in R\langle U_\mu, V_\mu, X, Y'\rangle$ and also a unit E of $R\langle U_\mu, V_\mu, X, Y\rangle$ such that*

$$F_\mu = E.(Y_n^\ell + c_1 Y_n^{\ell-1} + \cdots + c_\ell).$$

(iii) *Suppose $x \in R^m$ and $K \models S_\mu(x)$. Then, putting $u_{\mu\nu} := a_\nu(x)/a_\mu(x)$ (for $\nu \in A(\mu)$) and $v_{\mu\nu} := a_\nu(x)/t a_\mu(x)$ (for $\nu \in B(\mu)$), it follows that*

$$f(x, T_d(Y)) = a_\mu(x) F_\mu(x, Y) =$$

$$= a_\mu(x).E(u_\mu, v_\mu, x, Y).\big(Y_n^\ell + c_1(u_\mu, v_\mu, x, Y')Y_n^{\ell-1} + \cdots + c_\ell(u_\mu, v_\mu, x, Y')\big),$$

with $u_\mu = (u_{\mu\nu} : \nu \in A(\mu))$, $v_\mu = (v_{\mu\nu} : \nu \in B(\mu))$.

2.2 Holomorphic functions in elementary extensions

It is essential for our holomorphic functions to live in an algebraically closed extension field. Thus we fix from now on some extension field $\widehat{K}$ of K_1, algebraically closed and complete with respect to an absolute value extending that of K, with valuation ring $\widehat{R} = \{x \in \widehat{K} : |x| \leq 1\}$. (For instance, we could take for $\widehat{K}$ the completion of the algebraic closure of K.) Since $R\langle Y\rangle$ is a subring of $\widehat{R}\langle Y\rangle$, the language $\mathcal{L}_R$ is naturally a sublanguage of $\mathcal{L}_{\widehat{R}}$, so that $\widehat{K}$ is naturally an $\mathcal{L}_R$-structure, where we interpret the unary predicate symbol of $\mathcal{L}_R$ as $\widehat{R}$. We take the pair $(\widehat{K}, K)$ as our standard model, and view it as a structure for the language obtained from $\mathcal{L}_R$ by adding an extra unary predicate symbol Q, to be interpreted as the subset K of $\widehat{K}$.

Our main work takes place in an arbitrary elementary extension $(\widehat{K}_1, K_1)$, which we think of as our nonstandard model. Note that the interpretation of the unary predicate symbol of $\mathcal{L}_R$ in $(\widehat{K}_1, K_1)$ is a valuation ring $\widehat{R}_1$ of the field $\widehat{K}_1$. We denote the induced valuation on $\widehat{K}_1$ by v, and we let $R_1 := K_1 \cap \widehat{R}_1$ be the corresponding valuation ring of K_1. Because of its intuitive appeal we continue to write $|a| \leq |b|$ in place of $v(a) \geq v(b)$ (for $a, b \in \widehat{K}_1$) even though we do not attach a meaning to $|a|$ itself unless $a \in \widehat{K}$. Similarly we write $|a| = 1$ instead of $v(a) = 0$, and so on.

Given any tuple $U = (U_1, \ldots, U_M)$ of distinct indeterminates and any $f(U) \in R\langle U\rangle$, the function $u \mapsto f(u) : \widehat{R}^M \to \widehat{R}$ is definable in the structure $(\widehat{K}, K)$ and hence it has a canonical extension to a function $\widehat{R}_1^M \to \widehat{R}_1$ definable in $(\widehat{K}_1, K_1)$. By a slight abuse of notation, we will denote the value of this function at a point $u \in \widehat{R}_1^M$ also by $f(u)$. One should not think of $f(u)$ here as given by the corresponding convergent series as in the standard model; indeed the infinite sum may not converge in the topological field $\widehat{K}_1$. This explains the perhaps surprising length of the proof of Lemma 2.5 below.

Suppose now that $f(X, Y) = \sum_\nu a_\nu(X)Y^\nu \in R\langle X, Y\rangle$. For each $x \in R_1^m$, define

$$f(x, Y) := \sum a_\nu(x)Y^\nu \in R_1[[Y]].$$

By comparing coefficients of each term Y^ν, we can easily check that for any $x \in \widehat{R}_1^m$, the map $f \mapsto f(x, Y)$ is an R-algebra homomorphism $R\langle X, Y\rangle \to R_1[[Y]]$.

We now define

$$R_1\{Y\} := \{f(x, Y) : m \in \mathbf{N}, f \in R\langle X, Y\rangle, x \in R_1^m\},$$

$$K_1\{Y\} := \left\{c^{-1}g(Y) : c \in R_1 \setminus \{0\}, g(Y) \in R_1\{Y\}\right\} \subseteq K_1[[Y]].$$

Lemma 2.2 *If $g_1(Y), \ldots, g_k(Y) \in R_1\{Y\}$ then there is a single $m \in \mathbf{N}$, some $x \in R_1^m$, and $f_1, \ldots, f_k \in R\langle X, Y\rangle$ such that $g_i = f_i(x, Y)$ for each $i = 1, \ldots, k$.*

Proof. Introduce extra parametric variables and relabel if necessary. □

Corollary 2.3 *$R_1\{Y\}$ is a subring of $R_1[[Y]]$ which contains $R_1[Y]$.*

Proof. Observe that if $g_1(Y), g_2(Y) \in R_1\{Y\}$, then by Lemma 2.2 there is $m \in \mathbf{N}$ and $x \in R_1^m$ such that $g_i = f_i(x, Y)$ for $i = 1, 2$. Then

$$g_1(Y) + g_2(Y) = (f_1 + f_2)(x, Y),$$

$$g_1(Y).g_2(Y) = (f_1.f_2)(x, Y),$$

so $R_1\{Y\}$ is closed under addition and multiplication. □

Lemma 2.4 $K_1\{Y\} \cap R_1[[Y]] \subseteq R_1\{Y\}$.

Proof. Suppose $g(Y) = c^{-1}\sum a_\nu(x)Y^\nu \in K_1\{Y\} \cap R_1[[Y]]$, where $c \in R_1 \setminus \{0\}$ and $\sum a_\nu(X)Y^\nu \in R\langle X, Y\rangle$. As at the beginning of Section 2.1, there are $d \in \mathbf{N}$ and elements $b_{\nu\mu}(X) \in R\langle X\rangle$ such that for all ν with $|\nu| \le d$, $a_\nu(X) = \sum_{|\mu|<d} b_{\nu\mu}(X)a_\mu(X)$ and $||b_{\nu\mu}(X)|| \to 0$ as $\nu \to \infty$ for each fixed μ with $|\mu| < d$. Introduce new variables U_μ for $|\mu| < d$ and put $u_\mu = c^{-1}a_\mu(x) \in R_1$. Let

$$F(X, U, Y) = \sum_{|\nu|<d} U_\nu Y^\nu + \sum_{|\nu|\ge d} \Big(\sum_{|\mu|<d} b_{\nu\mu}(X)U_\mu\Big)Y^\nu.$$

Then $F(X, U, Y) \in R\langle X, U, Y\rangle$ and $g(Y) = F(x, u, Y)$. Thus $g(Y) \in R_1\{Y\}$. $\square$

The following lemma relates the power series $f(x, Y)$ to the corresponding function $\widehat{R}_1^n \to \widehat{R}_1$.

Lemma 2.5 *Let $x_0 \in R_1^m$. Then $f(x_0, Y) = 0$ (that is, $a_\nu(x_0) = 0$ for all $\nu \in \mathbf{N}^n$) if and only if $f(x_0, y) = 0$ for all $y \in \widehat{R}_1^n$.*

Proof. With d, the b_ν and f_μ as in Section 2.1, we have a finite sum

$$f(X, Y) = \sum_{|\mu|<d} a_\mu(X)f_\mu(X, Y).$$

So for all $x \in R^m$ and for all $y \in \widehat{R}^n$, we have $f(x, y) = \sum_{|\mu|<d} a_\mu(x)f_\mu(x, y)$, and hence this is also true for all $x \in R_1^m$ and $y \in \widehat{R}_1^n$.

Assume first that $f(x_0, Y) = 0$. Since $K_1 \models a_\nu(x_0) = 0$ for each ν, it follows that $f(x_0, y) = 0$ for all $y \in \widehat{R}_1^n$.

For the converse, suppose that $f(x_0, y) = 0$ for all $y \in \widehat{R}_1^n$. Now

$$K_1 \models (\forall x \in R_1^m)\big(Z(x) \vee \bigvee_{|\mu|<d} S_\mu(x)\big),$$

by Lemma 2.1. Also,

$$K \models (\forall x \in R^m)\big(Z(x) \to \bigwedge_{|\mu|<d} a_\mu(x) = 0\big),$$

hence for any $\nu \in \mathbf{N}^n$,

$$K \models (\forall x \in R^m)\big(Z(x) \to a_\nu(x) = 0\big).$$

It follows that if $K_1 \models Z(x_0)$, then $K_1 \models a_\nu(x_0) = 0$ for all $\nu \in \mathbf{N}^n$, so $f(x_0, Y) = 0$, as required. Suppose for a contradiction that $K_1 \models S_\mu(x_0)$ for some μ with $|\mu| < d$. Then $a_\mu(x_0) \neq 0$. By Lemma 2.1(iii) we have for all $y \in \widehat{R}_1^n$

$$f(x_0, T_d(y)) = a_\mu(x_0)E(u_\mu, v_\mu, x_0, y)\big(y_n^\ell + c_1(u_\mu, v_\mu, x_0, y')y_n^{\ell-1} + \cdots + c_\ell(u_\mu, v_\mu, x_0, y')\big).$$

Here, E is a unit of $R\langle U_\mu, V_\mu, X, Y\rangle$, so $E(u_\mu, v_\mu, x_0, y) \neq 0$ for all $y \in \widehat{R}_1^n$. Since $\widehat{R}_1$ is infinite, we can choose y so that the polynomial in y_n is non-zero. Then $f(x_0, T_d(y)) \neq 0$, which is a contradiction. $\square$

To every $g(Y) \in K_1\{Y\}$, we associate a function $\widehat{g} : \widehat{R}_1^n \to \widehat{K}_1$ as follows: choose $f \in R\langle X, Y\rangle$, $x \in R_1^m$ and $c \in R_1 \setminus \{0\}$ such that $g(Y) = c^{-1} f(x, Y)$, and set $\widehat{g}(y) := c^{-1} f(x, y)$ for all $y \in \widehat{R}_1^n$.

Lemma 2.6 **(i)** *$\widehat{g}$ does not depend on the choice of f, c and x; that is, if $c^{-1} f(x, Y) = {c'}^{-1} f'(x', Y)$, where $f \in R\langle X, Y\rangle$, $c \in R_1 \setminus \{0\}$, $X = (X_1, \ldots, X_m)$, $x \in R^m$ and $f' \in R\langle X', Y\rangle$, $c' \in R_1 \setminus \{0\}$, $X' = (X'_1, \ldots, {X'}_{m'})$, $x' \in R^{m'}$, then $c^{-1} f(x, y) = {c'}^{-1} f'(x', y)$ for all $y \in R^n$.*

(ii) *The map $g \mapsto \widehat{g}$ is an injective R_1-algebra homomorphism from $K_1\{Y\}$ to the ring of $\widehat{K}_1$-valued functions on $\widehat{R}_1^n$.*

Proof. (i) Suppose that $g(Y) = c^{-1} f(x, Y) = {c'}^{-1} f'(x', Y)$ as indicated, where

$$f(X, Y) = \sum a_\nu(X) Y^\nu \qquad \text{and} \qquad f'(X', Y) = \sum b_\nu(X') Y^\nu.$$

By cross-multiplying and absorbing c, c' into the coefficients we may reduce to the case that $c = c' = 1$, and by introducing extra parametric variables and relabelling, we may assume $m = m'$ and $x = x'$. Then for all $\nu \in \mathbf{N}^n$, $a_\nu(x) = b_\nu(x)$. Put

$$h(X, Y) := f(X, Y) - f'(X, Y) = \sum \big(a_\nu(X) - b_\nu(X)\big) Y^\nu.$$

Then $h(x, Y) = 0$, hence by Lemma 2.5, $h(x, y) = 0$ for all $y \in \widehat{R}_1^n$, so $f(x, y) = f'(x, y)$ for all $y \in \widehat{R}_1^n$, as required.

(ii) This is straightforward. Injectivity follows from Lemma 2.5. $\square$

From now on, we just write g for the corresponding function $\widehat{g}$. It follows from Lemma 2.6 that we can substitute elements of $R_1\{Z\}$ for the Y variables in $g(Y) \in K_1\{Y\}$. For suppose that $g \in K_1\{Y\}$, $h_1, \ldots, h_n \in R_1\{Z\}$, where $Z := (Z_1, \ldots, Z_N)$. Define $g(h_1, \ldots, h_n) \in K_1\{Z\}$ as follows. As in Lemma 2.2, there is $m \in \mathbf{N}$ and $x \in R_1^m$ such that we can write $g = c^{-1} G(x, Y)$, with $c \in R_1 \setminus \{0\}$ and $G(X, Y) \in R\langle X, Y\rangle$, and $h_i = H_i(x, Z)$ with $H_i(X, Y) \in R\langle X, Y\rangle$ for $i = 1, \ldots, n$. Put

$$F(X, Z) := G\big(X, H_1(X, Z), \ldots, H_n(X, Z)\big) \in R\langle X, Z\rangle.$$

Then $g(h_1, \ldots, h_n) := c^{-1} F(x, Z)$ is in $K_1\{Z\}$, and $g(h_1, \ldots, h_n)$ does not depend on the choice of G, $H_1, \ldots, H_n$ and x; moreover, $g(h_1, \ldots, h_n)(z) =$

$g(h_1(z), \ldots, h_n(z))$ for $z \in \widehat{R}_1^N$. It follows that any $\mathcal{L}_R$-term in Y with parameters from R_1 defines a function on $\widehat{R}_1^n$ given by some element of $R_1\{Y\}$. However, this does not hold for $\mathcal{L}_{\mathrm{an}}^D$-terms, and this motivates our covering of R_1 by connected affinoid sets in Section 4.

The following lemma is immediate from the remarks at the beginning of section 2.1.

Lemma 2.7 *Let $g(Y) = \sum c_\nu Y^\nu \in R_1\{Y\}$.*

(i) *There is an index μ such that for all $\nu > \mu$, $|c_\nu| < |c_\mu|$.*

(ii) *If $|c_\nu| < 1$ for all ν then $|g(y)| < 1$ for all $y \in \widehat{R}_1^n$.*

Next we prove Weierstrass preparation and division for $K_1\{Y\}$.

Proposition 2.8 *Let $g(Y) \in K_1\{Y\} \setminus \{0\}$ and $n > 0$. Then there are $d > 0$ and $\ell \in \mathbf{N}$ such that*

$$g(T_d(Y)) = c.E(Y).\big(Y_n^\ell + c_1(Y').Y_n^{\ell-1} + \cdots + c_\ell(Y')\big),$$

where $c \in K_1$, $E(Y) \in R_1\{Y\}$ is a unit, and $c_i(Y') \in R_1\{Y'\}$ for $i = 1, \ldots, \ell$. Moreover, each $h(Y) \in K_1\{Y\}$ can be written as

$$h(Y) = q(Y).g(T_d(Y)) + r(Y),$$

where $q(Y) \in K_1\{Y\}$ and $r(Y) \in K_1\{Y'\}[Y_n]$ is of degree less than ℓ in Y_n.

Proof. Let $g(Y) = a^{-1}f(x, Y)$, where $f(X, Y) \in R\langle X, Y\rangle$, $a \in R_1 \setminus \{0\}$. With the notation of Subsection 2.1, the proof of Lemma 2.5 gives $R_1 \models \neg Z(x)$, so there is $\mu \in \mathbf{N}^n$ with $|\mu| < d$ such that $R_1 \models S_\mu(x)$. By Lemma 2.1(iii), there is

$$\tilde{F}_\mu(U_\mu, V_\mu, X, Y) \in R\langle U_\mu, V_\mu, X, Y\rangle,$$

such that $F_\mu := \tilde{F}_\mu(U_\mu, V_\mu, X, T_d(Y))$ is regular in Y_n of degree ℓ (with ℓ as before), and

$$K_1 \models \forall\, y \in R_1^n \big(f(x, T_d(y)) = a_\mu(x) F_\mu(u_\mu(x), v_\mu(x), x, y)\big).$$

Furthermore, by Lemma 2.1(ii), there is a unit $E \in R\langle U_\mu, V_\mu, X, Y\rangle$ and elements $b_1, \ldots, b_\ell \in R\langle U_\mu, V_\mu, X, Y'\rangle$ such that

$$F_\mu = E.(Y_n^\ell + b_1 Y_n^{\ell-1} + \cdots + b_\ell).$$

It follows that

$$\begin{aligned} g(T_d(Y)) &= a^{-1} f(x, T_d(Y)) = a^{-1} a_\mu(x) F_\mu\big(u_\mu(x), v_\mu(x), x, T_d(Y)\big) \\ &= a^{-1} a_\mu(x) E(u_\mu, v_\mu, x, Y)\big(Y_n^\ell + b_1(u_\mu, v_\mu, x, Y') Y_n^{\ell-1} + \cdots + b_\ell(u_\mu, v_\mu, x, Y')\big). \end{aligned}$$

Now put $c := a^{-1}a_\mu(x)$ and $c_i(Y') := b_i(u_\mu(x), v_\mu(x), x, Y')$.

For the division theorem, note that by Lemma 2.2, we can assume $h(Y) = b^{-1}H(x, Y)$, with $H(X, Y) \in R\langle X, Y\rangle$, $b \in R_1 \setminus \{0\}$. By the division theorem for $R\langle U_\mu, V_\mu, X, Y\rangle$, there are $Q(U_\mu, V_\mu, X, Y) \in R\langle U_\mu, V_\mu, X, Y\rangle$ and $P(U_\mu, V_\mu, X, Y) \in R\langle U_\mu, V_\mu, X, Y'\rangle[Y_n]$ of degree less than ℓ such that

$$H(X, Y) = Q(U_\mu, V_\mu, X, Y).F_\mu(U_\mu, V_\mu, X, Y) + P(U_\mu, V_\mu, X, Y).$$

As before, substitution of x, u_μ, v_μ for X, U_μ, V_μ gives the desired statement. □

We emphasise that in the proof of Proposition 2.8, the number ℓ and also c, E and the c_i depend on which set S_μ contains x.

In the proof of Theorem A, we also need the following variant of Proposition 2.8.

Proposition 2.9 *Let $g(Y) \in K_1\{Y\}$, $n > 0$, be such that $g = g_1 + \cdots + g_n$, where $g_i(Y_i) \in K_1\{Y_i\}$ for each i and $g_i \neq 0$ for some i. Then there is $i \in \{1, \ldots, n\}$ and $\ell \in \mathbf{N}$ such that*

$$g(Y) = c.E(Y).\big(Y_i^\ell + c_1(Y^*)Y_i^{\ell-1} + \cdots + c_\ell(Y^*)\big),$$

where $c \in K_1$, $E(Y)$ is a unit of $R_1\{Y\}$, $Y^* := (Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$, and $c_i(Y^*) \in R_1\{Y^*\}$ for $i = 1, \ldots, \ell$.

Proof. Because there are no mixed terms in the Y_i, we do not need to apply a substitution $T_d(Y)$ for Y when arguing as in the proof of Proposition 2.8. After multiplying g by a suitable constant it is already regular in some variable. □

Lemma 2.10 *Let E be a unit of $R_1\{Y\}$. Then $|E(y)| = 1$ for all $y \in \widehat{R}_1^n$.*

Proof. There is $E^{-1} \in R_1\{Y\}$ such that $E(y).E^{-1}(y) = 1$, and $|E(y)| \leq 1$, $|E^{-1}(y)| \leq 1$ for all $y \in R_1^n$. The result follows. □

Finally we state some standard consequences of Proposition 2.8.

Corollary 2.11 **(i)** *$K_1\{Y\}$ is a noetherian domain.*

(ii) *For each proper ideal I of $K_1\{Y\}$ there is $e \leq n$ and an injective K_1-algebra homomorphism $K_1\{Y_1, \ldots, Y_e\} \to K_1\{Y\}/I$ making $K_1\{Y\}/I$ into a finitely generated module over $K_1\{Y_1, \ldots, Y_e\}$.*

(iii) *For each proper ideal I of $K_1\{Y\}$ there is $y \in \widehat{R}_1^n$ such that $f(y) = 0$ for all $f \in I$.*

Proof. First note that for each $d > 0$ the map $g(Y) \mapsto g(T_d(Y))$ is a K_1-algebra automorphism of $K_1\{Y\}$. Hence we can obtain (i) and (ii) just as the corresponding results II.3.1, II.3.2 and II.3.4 in [4] for the Tate ring. For (iii), one can use the proof of II.3.5 in [4] to first show that $K_1\{Y\}/M$ is a finite-dimensional vector space over K_1 for each maximal ideal M of $K_1\{Y\}$, and then show by induction on n that if M is a maximal ideal containing I and $\varphi : K_1\{Y\}/M \to \widehat{K}_1$ is any field embedding over K_1 and $y = (\varphi(Y_1 + M), \ldots, \varphi(Y_n + M))$ then $y \in \widehat{R}_1^n$ and $f(y) = 0$ for all $f \in M$. $\square$

3 Holomorphic functions on a connected affinoid

As we explained in the introduction, the proofs of our theorems make essential use of the properties of the rings $\mathcal{O}(F)$ of 'holomorphic' functions on 'connected affinoids' F. Roughly speaking, these affinoid algebras $\mathcal{O}(F)$ are introduced here as quotients of $K_1\{Y\}$. This enables us to derive their properties using results of the previous section. We keep the assumptions and notations from Section 2.

3.1 Connected affinoids

A *closed*, respectively *open*, R_1-*disk* is a subset of $\widehat{R}_1$ of the form $\{z \in \widehat{R}_1 : |z - a| \leq |\pi|\}$, respectively $\{z \in \widehat{R}_1 : |z - a| < |\pi|\}$, where $a, \pi \in R_1$ and $\pi \neq 0$. An R_1-*disk* is a closed R_1-disk or an open R_1-disk. We stress that these disks are subsets of $\widehat{R}_1$ rather than R_1, but they have 'center' and 'radius' defined by elements of R_1. We use tacitly the fact that each closed disk $\{z \in \widehat{R}_1 : |z - a| \leq |\pi|\}$ as well as each open disk $\{z \in \widehat{R}_1 : |z - a| < |\pi|\}$ with $a, \pi \in \widehat{R}_1$, $\pi \neq 0$, has either empty intersection with R_1 or the same intersection with R_1 as a closed R_1-disk. This is because each γ in the value group of $\widehat{K}_1$ has an 'integral part' in the value group of K_1 which is a $\mathbf{Z}$-group; that is, $\delta \leq \gamma < \delta + 1$ for some δ in the value group of K_1. A *connected* R_1-*affinoid* F is defined to be a subset of $\widehat{R}_1$ of the form $F = D \setminus (H_1 \cup \cdots \cup H_k)$ where D is a closed R_1-disk and $H_1, \ldots, H_k$, $k \geq 0$, are disjoint open R_1-disks contained in D. It is easy to check that F uniquely determines D and the set $\{H_1, \ldots, H_k\}$ of its 'holes'. It is also easy to see that if F_1 and F_2 are connected R_1-affinoids and $F_1 \cap F_2 \neq \emptyset$ then $F_1 \cup F_2$ and $F_1 \cap F_2$ are also connected R_1-affinoids. (Use that if A and B are R_1-disks, then either $A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$.) We call attention to the possibility that, although a connected R_1-affinoid is always infinite, its intersection with R_1 may be empty. The next lemma states two further basic properties of connected R_1-affinoids that will be needed later.

Lemma 3.1 **(i)** *Given a finite collection $\mathcal{C}$ of connected R_1-affinoids, there is a finite collection $\mathcal{F}$ of pairwise disjoint connected R_1-affinoids whose union* *contains R_1 (but not necessarily $\widehat{R}_1$) and which refines $\mathcal{C}$ (that is, if $C \in \mathcal{C}$, $F \in \mathcal{F}$ and $C \cap F \neq \emptyset$ then $F \subseteq C$).*

(ii) *Let F be a connected R_1-affinoid and let $r_1, r_2 \in K_1(Z)$ be two rational functions that have no poles in F. Then there is a finite set $E \subset F$ and there are connected R_1-affinoids $F_1, \ldots, F_k \subseteq F$, with $k \geq 0$, such that $|r_1(z)| \leq |r_2(z)|$ for all $z \in F_1 \cup \cdots \cup F_k$ and*

$$\{z \in F \cap R_1 : |r_1(z)| \leq |r_2(z)|\} \subseteq E \cup F_1 \cup \cdots \cup F_k.$$

If $r_2 = 1$ we can take $E = \emptyset$.

Proof. We leave (i) as an exercise to the reader. For (ii) we note that the result is obvious if $r_1 = 0$ or $r_2 = 0$. So assume $r_1 \neq 0$ and $r_2 \neq 0$, and let E be the set of common zeros of r_1 and r_2 on F. One can argue as in [4, I.1.2] to show that $\{z \in F : |r_1(z)| \leq |r_2(z)|\}$ is the union of E and finitely many sets of the form $A \setminus (B_1 \cup \cdots \cup B_m)$ with A a closed disk in $\widehat{R}_1$ (not necessarily a closed R_1-disk) and $B_1, \ldots, B_m \subseteq A$ open disks in $\widehat{R}_1$ (not necessarily open R_1-disks). One easily verifies that the intersection of such a set $A \setminus (B_1 \cup \cdots \cup B_m)$ with R_1 is empty or of the form $F' \cap R_1$ where F' is a connected R_1-affinoid contained in $A \setminus (B_1 \cup \cdots \cup B_m)$. This gives the desired result. □

3.2 The affinoid algebra associated to a connected affinoid.

Let F be a connected R_1-affinoid. Then there are $n \geq 1$ and $a_1, \ldots, a_n, \pi_1, \ldots, \pi_n \in R_1$ with $\pi_1 \neq 0, \ldots, \pi_n \neq 0$ such that

$$F = \{z \in \widehat{R}_1 : |z - a_1| \leq |\pi_1| \wedge \bigwedge_{i=2}^{n} |z - a_i| \geq |\pi_i|\},$$

where the open R_1-disks $\{z \in \widehat{R}_1 : |z - a_i| < |\pi_i|\}$, for $2 \leq i \leq n$, are mutually disjoint, and are all contained in the closed R_1-disk $\{z \in \widehat{R}_1 : |z - a_1| \leq |\pi_1|\}$. In this situation we will say that F is *given by* $(a_1, \ldots, a_n; \pi_1, \ldots, \pi_n)$. Define $\psi : F \to \widehat{R}_1^n$ by

$$\psi(z) = (\frac{z - a_1}{\pi_1}, \frac{\pi_2}{z - a_2}, \ldots, \frac{\pi_n}{z - a_n}).$$

For distinct $i, j \in \{2, \ldots, n\}$, define $s_{ij} := \frac{\pi_i}{a_i - a_j}$, and for each $j = 2, \ldots, n$, let $s_{1j} = \frac{a_j - a_1}{\pi_1}$, and $s_{j1} := \frac{\pi_j}{\pi_1}$. Notice that $|s_{ij}| \leq 1$. Also, for $2 \leq i < j \leq n$, define

$$e_{ij}(Y) := s_{ij} Y_j + s_{ji} Y_i + Y_i Y_j,$$

and for $j = 2, \ldots, n$ put

$$e_{1j}(Y) := Y_j(Y_1 - s_{1j}) - s_{j1}.$$

Let $I(F)$ be the ideal of $K_1\{Y\}$ generated by the e_{ij} with $1 \leq i < j \leq n$. Then

$$\psi(F) = V\big(I(F)\big) := \{y \in R_1^n : \text{ for all } f \in I(F), f(y) = 0\}.$$

To see the inclusion from right to left, let $y = (y_1, \ldots, y_n) \in V(I(F))$. Since $y_1 \in R_1$, we may put $y_1 = \frac{z-a_1}{\pi_1}$ for some $z \in R_1$ with $|z - a_1| \leq \pi_1$. For each $i = 2, \ldots, n$, since $e_{1i}(y) = 0$ we have $y_i = \frac{\pi_i}{z-a_i}$. From $y_i \in R_1$ for each i, it follows that $z \in F$, and hence that $y \in \psi(F)$, as required. The next two lemmas show that elements of $K_1\{Y\}$ have a fairly simple form modulo $I(F)$.

Lemma 3.2 **(i)** *Let $g \in K_1\{Y\}$. Then there are $g_i(Y_i) \in K_1\{Y_i\}$ (for $i = 1, \ldots, n$) such that $g - (g_1 + \ldots + g_n) \in I(F)$.*

(ii) *Let $g \in K_1\{Y_2, \ldots, Y_n\}$. Then there are $g_i(Y_i) \in K_1\{Y_i\}$ (for $i = 2, \ldots, n$) such that, if I_1 is the ideal in $K_1\{Y_2, \ldots, Y_n\}$ generated by the e_{ij} for $2 \leq i < j \leq n$, then $g - (g_2 + \cdots + g_n) \in I_1$.*

Proof. (i) We use induction on n. In the case $n = 1$, there is nothing to prove, so assume that $n > 1$. Let F' be the connected R_1-affinoid given by $(a_1, \ldots, a_{n-1}; \pi_1, \ldots, \pi_{n-1})$ and let $I(F')$ be the corresponding ideal of $K_1\{Y'\}$ generated by the e_{ij} with $1 \leq i < j \leq n-1$. Suppose for the inductive hypothesis that each $g' \in K\{Y'\}$ is congruent modulo $I(F')$ to a sum $g'_1 + \cdots + g'_{n-1}$ with $g'_i = g'_i(Y_i) \in K_1\{Y_i\}$. It is clear that the result follows from the following claim.

Claim. There is $h_1 \in K_1\{Y'\}$ and $h_2 \in K_1\{Y_n\}$ such that $g - (h_1 + h_2) \in I(F)$.

Proof. After multiplying g by a constant if necessary, we may suppose that $g \in R_1\{Y\}$; say $g(Y) = G(x, Y)$, where $G(X, Y) = \sum_\nu a_\nu(X)Y^\nu \in R\langle X, Y\rangle$ and $x \in R_1^m$. Let

$$S := (S_{1n}, S_{n1}, \ldots, S_{in}, S_{ni}, \ldots, S_{n-1,n}, S_{n,n-1})$$

be a sequence of indeterminates. Define

$$E_{in} = S_{in}Y_n + S_{ni}Y_i + Y_iY_n$$

for $i \in \{2, \ldots, n-1\}$, and

$$E_{1n} = Y_n(Y_1 - S_{1n}) - S_{n1}.$$

A short inductive argument shows that for any $\nu \in \mathbf{N}^n$ there are polynomials $p^{(\nu)}(S, Y') \in R[S, Y']$ and $q^{(\nu)}(S, Y_n) \in R[S, Y_n]$ of total degree at most $|\nu|$ and $r_i^{(\nu)}(S, Y) \in R[S, Y]$, for $1 \leq i \leq n-1$, of total degree at most $|\nu| - 2$ such that

$$Y^\nu = p^{(\nu)}(S, Y') + q^{(\nu)}(S, Y_n) + \sum_{i=1}^{n-1} r_i^{(\nu)}(S, Y)E_{in}.$$

Furthermore, the least degree of a monomial in $p^{(\nu)}$ tends to infinity with $|\nu|$, and the same holds with $q^{(\nu)}$ and the $r_i^{(\nu)}$. Put

$$h_1(X,S,Y') = \sum_\nu a_\nu(X)p^{(\nu)}(S,Y'),$$

$$h_2(X,S,Y_n) = \sum_\nu a_\nu(X)q^{(\nu)}(S,Y_n), \text{ and}$$

$$r_i(X,S,Y) = \sum_\nu a_\nu(X)r_i^{(\nu)}(S,Y),$$

for $i = 1,\ldots,n-1$. By the above observation on the minimal degrees of monomials in the $p^{(\nu)}$, $q^{(\nu)}$ and $r_i^{(\nu)}$, we have

$$h_1(X,S,Y'), h_2(X,S,Y_n), r_i(X,S,Y) \in R\langle X,S,Y\rangle.$$

Then

$$\sum_\nu a_\nu(X)Y^\nu = h_1(X,S,Y') + h_2(X,S,Y_n) + \sum_{i=1}^{n-1} E_{in}r_i(X,S,Y).$$

Let $s = (s_{1n}, s_{n1}, \ldots, s_{in}, s_{ni}, \ldots, s_{n-1,n}, s_{n,n-1})$, so $e_{in} = E_{in}(s,Y)$ for $1 \le i < n$. Thus,

$$g(Y) = G(x,Y) = h_1(x,s,Y') + h_2(x,s,Y_n) + E(x,s,Y),$$

where $E(x,s,Y) \in I(F)$. The claim follows.

The proof of (ii) is a simpler version of that of (i), so is omitted. $\square$

Proposition 3.3 *Let $g \in K_1\{Y\}$, and let F be a connected R_1-affinoid given by $(a_1,\ldots,a_n;\pi_1,\ldots,\pi_n)$. There are a unit $E \in R_1\{Y\}$, a polynomial $q(Y_k) \in K_1[Y_k]$ for some $k \in \{1,\ldots,n\}$, and a polynomial $p(Y) \in R_1[Y]$ which is a product of finitely many factors, each of which is of the form $(Y_j + s_{ji})$, with $2 \le i,j \le n$, $i \ne j$, or of the form $(Y_1 - s_{1i})$, with $2 \le i \le n$, or of the form Y_i, with $2 \le i \le n$, such that $pg - Eq \in I(F)$.*

Proof. By Lemma 3.2, we may suppose that there are $g_i \in K_1\{Y_i\}$ (for $i = 1,\ldots,n$) such that $g = g_1 + \cdots + g_n$ modulo $I(F)$. By Proposition 2.9, there are $i \in \{1,\ldots,n\}$, $\ell \in \mathbf{N}$, $c \in K_1$, a unit E of $R_1\{Y\}$, and $c_1,\ldots,c_\ell \in R_1\{Y^*\}$ (where $Y^* := (Y_1,\ldots,Y_{i-1},Y_{i+1},\ldots,Y_n)$) such that

$$g(Y) = c.E(Y).\big(Y_i^\ell + c_1Y_i^{\ell-1} + \cdots + c_\ell\big).$$

If $n = 1$ the desired property holds with $p(Y) = 1$. Suppose $n > 1$ in what follows. Pick $j \in \{1,\ldots,n\} \setminus \{i\}$, and suppose that $i,j > 1$ for the moment.

Since $e_{ij} - s_{ij}Y_j = Y_i(s_{ji} + Y_j)$, we can multiply throughout by $(s_{ji} + Y_j)^\ell$ and simplify to get

$$\begin{aligned} g(Y)(s_{ji}+Y_j)^\ell &= cE(Y)\big((e_{ij}-s_{ij}Y_j)^\ell + c_1(Y^*)(e_{ij}-s_{ij}Y_j)^{\ell-1}(s_{ji}+Y_j)+\cdots \\ &\qquad +c_\ell(Y^*)(s_{ji}+Y_j)^\ell\big). \end{aligned}$$

Since the e_{ij} are in $I(F)$, if

$$h(Y^*) := (-s_{ij}Y_j)^\ell + c_1(Y^*)(-s_{ij}Y_j)^{\ell-1}(s_{ji}+Y_j) + \cdots + c_\ell(Y^*)(s_{ji}+Y_j)^\ell,$$

we have

$$(s_{ji}+Y_j)^\ell.g - cE.h(Y^*) \in I(F).$$

If $j = 1$ the identity $e_{1i} + s_{i1} = Y_i(Y_1 - s_{1i})$ introduces a factor $(Y_1 - s_{1i})^\ell$, and if $i = 1$ we get similarly a factor Y_j^ℓ. We now apply a similar argument to $h(Y^*)$, noting that the new Y_i and Y_j must be among the remaining variables in Y^*. After at most $n-1$ repetitions, we obtain a polynomial $p(Y)$, a unit E of $R_1\{Y\}$, and a polynomial $q(Y_k)$ for some k with $1 \le k \le n$ such that

$$p(Y)g(Y) - Eq(Y_k) \in I(F).$$

$\square$

Let $\mathcal{R}(F)$ be the ring of all $\widehat{K}_1$-valued functions on F. Let $\psi^* : K_1\{Y\} \to \mathcal{R}(F)$ be the ring homomorphism given by $\psi^*(g) = g \circ \psi$ and define the *affinoid algebra of F*, $\mathcal{O}(F)$, to be the image of $K_1\{Y\}$ under ψ^*. Then $\mathcal{O}(F)$ is a K_1-subalgebra of $\mathcal{R}(F)$, and $I(F) = \ker(\psi^*)$: the inclusion $I(F) \subseteq \ker(\psi^*)$ is clear, and the reverse inclusion is shown in Corollary 3.8. But first we derive from Proposition 3.3 that elements of $O(F)$ can be written in a simple form.

Corollary 3.4 *Let F be a connected R_1-affinoid and let $f \in \mathcal{O}(F)$. There is a unit $E(Y) \in R_1\{Y\}$ and a rational function $r(Z) \in K_1(Z)$ with no poles in F such that for all $z \in F$, $f(z) = \psi^*(E)(z).r(z)$.*

Proof. Let $g \in K_1\{Y\}$ be such that $f = \psi^*(g)$. Let $E(Y) \in R_1\{Y\}$, $q(Y_k) \in K_1[Y_k]$ and $p(Y) \in R_1[Y]$ be as given in Proposition 3.3. We have

$$\begin{aligned} \psi^*(Y_j+s_{ji})(z) &= \frac{\pi_j(z-a_i)}{(a_j-a_i)(z-a_j)} \qquad \text{for } 2 \le i,j \le n,\ i \ne j, \\ \psi^*(Y_1-s_{1i})(z) &= \frac{z-a_i}{\pi_1} \qquad \text{for } 2 \le i \le n, \\ \psi^*(Y_i)(z) &= \frac{\pi_i}{z-a_i} \qquad \text{for } 2 \le i \le n. \end{aligned}$$

Clearly, $(\psi^*(p))(z) \neq 0$ for all $z \in F$. If we let $r(z)$ be the rational function

$$\frac{(\psi^*(q))(z)}{(\psi^*(p))(z)},$$

we obtain the required result. □

The definition of $\mathcal{O}(F)$ as stated depends on the choice of $(a_1, \ldots, a_n; \pi_1, \ldots, \pi_n)$ used to give F as a connected R_1-affinoid. The next lemma shows that $\mathcal{O}(F)$ is independent of this choice.

Lemma 3.5 *Let F be a connected R_1-affinoid given by $(a_1, \ldots, a_n; \pi_1, \ldots, \pi_n)$. Then the ring $\mathcal{O}(F)$ is independent of the choice of $(a_1, \ldots, a_n, \pi_1, \ldots, \pi_n)$.*

Proof. Suppose that F is also given by $(a'_1, \ldots, a'_n; \pi'_1, \ldots, \pi'_n)$. We first consider the case that $a_1 = a'_1$, $\pi_1 = \pi'_1$ and there is a permutation λ of $\{2, \ldots, n\}$ such that $a'_{\lambda(i)} = a_i$ and $\pi'_{\lambda(i)} = \pi_i$ for $2 \leq i \leq n$. Let $\psi' : F \to \widehat{R}_1^n$ be the map associated to $(a'_1, \ldots, a'_n; \pi'_1, \ldots, \pi'_n)$. Then clearly $\psi(z) = \lambda(\psi'(z))$, where for $y \in \widehat{R}_1^n$ we put $\lambda(y) = (y_1, y_{\lambda(2)}, \ldots, y_{\lambda(n)})$. It follows immediately that $\mathcal{O}(F)$ as defined via ψ is the same as when defined via ψ'. Using this special case, we may suppose for the general case that for each i the R_1-disks

$$\{z : |z - a_i| < |\pi_i|\} \text{ and } \{z : |z - a'_i| < |\pi'_i|\}$$

are equal. Let $f \in \mathcal{O}(F)$, let

$$G(X, Y) = \sum a_\nu(X) Y^\nu \in R\langle X, Y\rangle$$

and let $x \in R_1^m$, $c \in R_1$ be such that $f = \psi^*(c^{-1}g)$, where $g(Y) = G(x, Y)$. Define $v_1 := \frac{\pi'_1}{\pi_1}$, $u_1 := \frac{a_1 - a'_1}{\pi'_1}$, $v_i := \frac{\pi_i}{\pi'_i}$ and $u_i := \frac{a_i - a'_i}{t\pi'_i}$, for $i = 2, \ldots, n$, and introduce corresponding indeterminates $V_1, \ldots, V_n, U_1, \ldots, U_n$. We have

$$\psi^*(Y_1)(z) = \frac{z - a_1}{\pi_1} = v_1\big(\frac{z - a'_1}{\pi'_1} - u_1\big),$$

and

$$\psi^*(Y_i)(z) = \frac{\pi_i}{z - a_i} = \frac{\pi_i}{(z - a'_i)\big(1 - \frac{(a_i - a'_i)}{(z - a'_i)}\big)} = v_i \sum_{k=0}^{\infty} t^k u_i^k \left(\frac{\pi'_i}{z - a'_i}\right)^{k+1},$$

for $z \in F$, where the last infinite sum is to be interpreted as $Q(u_i, \pi'_i/(z - a'_i))$ for the obvious power series $Q(U_i, W_i) \in R\langle U_i, W_i\rangle$. Let $W = (W_1, \ldots, W_n)$

and let $\psi'^* : K_1\{W\} \to \mathcal{R}(F)$ be induced by the map $W_1 \mapsto \frac{z-a_1'}{\pi_1'}$, $W_i \mapsto \frac{\pi_i'}{z-a_i'}$, for $i = 2, \dots, n$. Define $H \in R\langle X, U, V, W\rangle$ to be

$$G\big(X, V_1(W_1 - U_1), V_2 \sum_{k=0}^{\infty} (tU_2)^k W_2^{k+1}, \dots, V_n \sum_{k=0}^{\infty} (tU_n)^k W_n^{k+1}\big).$$

Put $h(W) = H(x, u, v, W) \in R_1\{W\}$. It is easily checked with a transfer argument that $\big(\psi'^*(c^{-1}h)\big)(z) = f(z)$ for all $z \in F$, as required. $\square$

Lemma 3.6 *Let F, F' be connected R_1-affinoids with $F' \subset F$. Then for all $f \in \mathcal{O}(F)$, $f\,|_{F'} \in \mathcal{O}(F')$.*

Proof. This is like the proof of Lemma 3.5. Let F be given as usual and suppose for example that F' is given by $(a_1, a, a_{r+1} \dots, a_n; \pi_1, \pi, \pi_{r+1} \dots, \pi_n)$, where $a, \pi \in R_1$ are such that $\{z \in \widehat{R}_1 : |z - a| < |\pi|\}$ contains $\{z \in \widehat{R}_1 : |z - a_i| < |\pi_i|\}$ for $i = 2, \dots, r$ and is disjoint from $\{z \in \widehat{R}_1 : |z - a_i| < |\pi_i|\}$ for $i = r+1, \dots, n$. Let $f \in \mathcal{O}(F)$, and let $g \in R_1\{Y\}$ and $c \in R$ be such that $\psi^*(c^{-1}g) = f$. Let $W = (W_1, \dots, W_{n-(r-2)})$ be a sequence of indeterminates and let $\psi'^* : K_1\{W\} \to \mathcal{O}(F')$ be the map associated to $(a_1, a, a_{r+1} \dots, a_n; \pi_1, \pi, \pi_{r+1} \dots, \pi_n)$. Suppose that $g(Y) = G(x, Y) = \sum_\nu a_\nu(x) Y^\nu$. Define $u_i := \frac{a_i - a}{t\pi}$ and $v_i := \frac{\pi_i}{\pi}$ for $i = 2, \dots, r$ and introduce corresponding indeterminates $U = (U_2, \dots, U_r)$ and $V = (V_2, \dots, V_r)$. Define $H \in R\langle X, U, V, W\rangle$ to be

$$G(X, W_1, V_2 \sum_{k=0}^{\infty} (tU_2)^k W_2^{k+1}, \dots, V_r \sum_{k=0}^{\infty} (tU_r)^k W_2^{k+1}, W_3, \dots, W_{n-(r-2)})$$

Put $h(W) := H(x, u, v, W)$. Then $\psi'^*(c^{-1}h)\,|_{F'} = f\,|_{F'}$, as required.

The other cases to consider are when $n = 1$, or when π_1 is replaced by something of smaller norm to obtain F'. We omit the details, as they are similar. $\square$

Proposition 3.7 *Let F be a connected R_1-affinoid and $f \in \mathcal{O}(F)$ such that $|f(z)| \le 1$ for all $z \in F$. Then $f = \psi^*(g)$ for some $g \in R_1\{Y\}$.*

Proof. Let F be given by $(a_1, \dots, a_n; \pi_1, \dots, \pi_n)$. We have $f = \psi^*(c^{-1}g)$ for some $g \in R_1\{Y\}$ and $c \in R_1 \setminus \{0\}$. By Lemma 3.2, we can assume that $g = g_1 + \cdots + g_n$, where $g_i = g_i(Y_i) \in R_1\{Y_i\}$ and (using Lemmas 2.4 and 2.7(i)) at least one of the g_i has a coefficient of norm 1. Furthermore, we can assume $g_i(0) = 0$ for $i = 2, \dots, n$. We show that there is $z \in F$ such that $|g(\psi(z))| = 1$ and hence $1 \ge |f(z)| = |c^{-1}g(\psi(z))| = |c^{-1}|$, so $c^{-1} \in R_1$. Write $g_1(Y_1) = \sum_{k\ge 0} a_{1k}(x) Y_1^k$, $g_i(Y_i) = \sum_{k \ge 1} a_{ik}(x) Y_i^k$, for $2 \le i \le n$, where $x \in R_1^m$ and $\sum a_{ik}(X) Y_i^k \in R\langle X, Y\rangle$ for $1 \le i \le n$. We argue as in [4, I.1.3].

Case 1. $|a_{1k}(x)| = 1$ for some k. After rearranging $a_2, \ldots, a_n$, we may assume that for some $s \in \{1, \ldots, n\}$ and for all $i \in \{2, \ldots, s\}$, we have $|a_{ik}(x)| = 1$ for some k and $|\pi_i| = |\pi_1|$, whilst for all $i \in \{s+1, \ldots, n\}$, either $|a_{ik}(x)| < 1$ for all k or $|\pi_i| < |\pi_1|$. Suppose $z \in F$ is such that $|z - a_i| = |\pi_1|$ for all $i \in \{2, \ldots, n\}$, and let $y = \psi(z)$. If $i > s$ and $|\pi_i| < |\pi_1|$ then $|y_i| < 1$. So in any case, if $i \in \{s+1, \ldots, n\}$ then $|g_i(y_i)| < 1$ by Lemma 2.7(ii). Let $h(Y) := \sum_{i=1}^{s} g_i(Y_i)$. By Lemma 2.7(i), we can write $h(Y) = h_1(Y) + h_2(Y)$ where $h_1(Y) \in R_1[Y]$ and $h_2(Y) \in R_1\{Y\}$ has all its coefficients with norm less than 1. By Lemma 2.7(ii), $|g(y)| = |h(y)| = |h_1(y)| = 1$ for all $y \in R_1^n$ such that $|h_1(y)| = 1$. Define $k(W)$ to be the function

$$\begin{aligned} k(W) &:= h_1(W, \frac{\pi_2}{\pi_1 W + a_1 - a_2}, \cdots, \frac{\pi_n}{\pi_1 W + a_1 - a_n}) \\ &= \sum_{k=0}^{M_0} a_{1k}(x) W^k + \sum_{i=2}^{s} \sum_{k=1}^{M_i} a_{ik}(x) \frac{c_{ik}}{(W - d_i)^k}, \end{aligned}$$

where $c_{ik} = (\pi_i/\pi_1)^k$ and $d_i = (a_i - a_1)/\pi_1$. Then $|c_{ik}| = 1$ and $|d_i - d_j| = 1$ for $2 \le i < j \le s$. If $\bar{k}(W)$ is the image of k in the residue field, then each of the summands of $\bar{k}$ is not identically zero, the $\bar{d}_i$ are distinct and hence $\bar{k}$ is a nonzero rational function (clear denominators and substitute $W = d_2$). So there are infinitely many values $w \in R_1$ such that $\bar{k}(\bar{w}) \neq 0$ and hence there will be $z = \pi_1 w + a_1 \in F$ satisfying the conditions above such that $|g(\psi(z)| = |h(\psi(z))| = |k(w)| = 1$.

Case 2. $|a_{1k}(x)| < 1$ for all k. After rearranging $a_2, \ldots, a_n$, we may assume that $|a_{2k}(x)| = 1$ for some k and $|\pi_2| \ge |\pi_i|$ for all $i \in \{2, \ldots, n\}$ such that $|a_{ik}(x)| = 1$ for some k. Also, we may assume there is $s \in \{2, \ldots, n\}$ such that for all $i \in \{2, \ldots, s\}$, we have $|a_{ik}(x)| = 1$ for some k and $|a_2 - a_i| = |\pi_i|$, and for all $i \in \{s+1, \ldots, n\}$ we have either $|a_{ik}(x)| < 1$ for all k or $|a_2 - a_i| > |\pi_2| \ge |\pi_i|$, or $|a_2 - a_i| = |\pi_2| > |\pi_i|$. We consider $z \in F$ such that $|z - a_i| = |\pi_2|$ for all $i \in \{2, \ldots, s\}$ for which $|a_2 - a_i| = |\pi_2|$. Then the argument is as in Case 1. □

Corollary 3.8 *Let F be a connected R_1-affinoid. Then $I(F) = \ker(\psi^*)$, so in particular $\mathcal{O}(F) \simeq K_1\{Y\}/I(F)$.*

Proof. Let $g \in \ker(\psi^*)$. To show that $g \in I(F)$ we may, by Lemma 3.2, reduce to the case that $g = g_1 + \cdots + g_n$ with $g_i = g_i(Y_i) \in K_1\{Y_i\}$. As in the proof of Proposition 3.7, we may further assume that $g_i(0) = 0$ for $i = 2, \ldots, n$. Then the proof of 3.7 shows that if some $g_i \neq 0$ then $g(\psi(z)) \neq 0$ for some $z \in F$, which would contradict the assumption on g. □

3.3 Divisibility in $\mathcal{O}(F)$.

Theorem 3.9 *Let F be a connected R_1-affinoid and let f be a non-zero element of $\mathcal{O}(F)$.*

(i) *There are just finitely many zeros of f in F.*

(ii) *If f has no zeros in F then f is a unit of $\mathcal{O}(F)$.*

Proof. (i) This follows immediately from Corollary 3.4.

(ii) Let $g(Y) \in K_1\{Y\}$ be such that $f = \psi^*(g)$. Suppose that f is not a unit in $\mathcal{O}(F)$. Then $1 \notin gK_1\{Y\} + I(F)$, as $I(F) = \ker(\psi^*)$, hence by Corollary 2.11(iii), there is $y \in \widehat{R}_1^n$ such that $g(y) = 0$ and $h(y) = 0$ for all $h \in I(F)$. So $y \in V(I(F)) = \psi(F)$ by the discussion in the beginning of section 3.2, that is, $y = \psi(z)$ for some $z \in F$, which gives $f(z) = g(y) = 0$. $\square$

Corollary 3.10 *Let F be a connected R_1-affinoid, f_1 and f_2 elements of $\mathcal{O}(F)$.*

(i) *If $|f_1(z)| \leq |f_2(z)|$ for all $z \in F$ then f_2 divides f_1 in $\mathcal{O}(F)$.*

(ii) *There is a finite set $E \subset F$ and connected R_1-affinoids $F_1, \dots, F_k \subseteq F$ such that $|f_1(z)| \leq |f_2(z)|$ for all $z \in F_1 \cup \dots \cup F_k$ and*

$$\{z \in F \cap R_1 : |f_1(z)| \leq |f_2(z)|\} \subseteq E \cup F_1 \cup \dots \cup F_k \ .$$

Moreover, if $f_2 = 1$ we can take $E = \emptyset$.

Proof. Let $E_1(Y), E_2(Y) \in R_1\{Y\}$ and $r_1(Z), r_2(Z) \in K_1(Z)$ be as given by Corollary 3.4 for f_1 and f_2. We may assume $r_1 \neq 0$ and $r_2 \neq 0$, as otherwise the result is trivial. Since E_1 and E_2 are units, $|\psi^*(E_1)(z)| = |\psi^*(E_2)(z)| = 1$ for all $z \in F$.

(i) The hypothesis implies that $|r_1(z)| \leq |r_2(z)|$ for all $z \in F$. Clearly, if $\alpha \in F$ is a zero of r_2 of order ℓ, then α is also a zero of r_1 of at least order ℓ. Thus r_1/r_2 is a rational function with no poles in F. Hence by Theorem 3.9, when considered as a function on F, it belongs to $\mathcal{O}(F)$. Thus $f_1/f_2 = \psi^*(E_1)\psi^*({E_2}^{-1})r_1/r_2$ is in $\mathcal{O}(F)$.

(ii) By the above remarks, $\{z \in F : |f_1(z)| \leq |f_2(z)|\} = \{z \in F : |r_1(z)| \leq |r_2(z)|\}$. Apply Lemma 3.1(ii). $\square$

We include the next result for completeness; it is not needed later.

Proposition 3.11 *Let F be a connected R_1-affinoid. Then $\mathcal{O}(F)$ is a principal ideal domain.*

Proof. This is immediate from the fact that each nonzero element of $\mathcal{O}(F)$ is a product of a unit and a rational function without poles in $\mathcal{O}(F)$. $\square$

4 Proof of Theorem A′.

We now consider the special case where K is the field $\mathbf{Q}_p$ (so $R = \mathbf{Z}_p$, and we may assume $t = p$). For convenience we still use the notation, $K, R, K_1, R_1, \widehat{K}, \widehat{K_1}$, and so on.

Proposition 4.1 *Let $\tau(Z)$ be an $\mathcal{L}_{\mathrm{an}}^D$-term with parameters from R_1. Then there are disjoint connected R_1-affinoids $F_1, \ldots, F_k$ and functions $f_i \in \mathcal{O}(F_i)$ such that $R_1 \subseteq F_1 \cup \cdots \cup F_k$ and $\tau(z) = f_i(z)$ for all but finitely many $z \in F_i \cap R_1$, for $i = 1, \ldots, k$.*

Proof. By induction on the complexity of τ. If τ is a constant from R_1 or $\tau(Z) = Z$ then the result is immediate. If $\tau(Z) = \tau_1(Z) + \tau_2(Z)$ or $\tau(Z) = \tau_1(Z) \cdot \tau_2(Z)$, and the result holds for the terms τ_1, τ_2 then the desired result follows by an application of Lemma 3.1(i). Suppose now that $\tau(Z) = g(\tau_1(Z), \ldots, \tau_n(Z))$, where $g \in R\langle Y \rangle$, and assume inductively that the result holds for the terms $\tau_1, \ldots, \tau_n$. Note that $\tau(z) = 0$ whenever $z \in R_1$ and $|\tau_\ell(z)| > 1$ for some ℓ. By Corollary 3.10(ii) and the inductive hypothesis there are disjoint connected R_1-affinoids $F_1, \ldots, F_k$ such that $R_1 \subseteq F_1 \cup \cdots \cup F_k$ and such that for each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$, either $|\tau_j(z)| > 1$ for all but finitely many $z \in F_i \cap R_1$ or there is $f_{ij} \in \mathcal{O}(F_i)$ with $|f_{ij}(z)| \leq 1$ for all $z \in F_i$ and $\tau_j(z) = f_{ij}(z)$ for all but finitely many $z \in F_i \cap R_1$. Using Proposition 3.7 and taking into account the remark following Lemma 2.6 it follows that the desired result holds for τ and these $F_1, \ldots, F_k$.

Finally, suppose $\tau(Z) = D(\tau_1(Z), \tau_2(Z))$ and assume inductively that the result holds for τ_1, τ_2. As in the previous case we obtain disjoint connected R_1-affinoids $F_1, \ldots, F_k$ such that $R_1 \subseteq F_1 \cup \cdots \cup F_k$ and such that for each $i \in \{1, \ldots, k\}$, either $|\tau_1(z)| > 1$ for all but finitely many $z \in F_i \cap R_1$, or $|\tau_2(z)| > 1$ for all but finitely many $z \in F_i \cap R_1$, or there are $f_{ij} \in \mathcal{O}(F_i)$ for $j = 1, 2$ with the following properties:

(a) $|f_{ij}(z)| \leq 1$ for all $z \in F_i$ and $\tau_j(z) = f_{ij}(z)$ for all but finitely many $z \in F_i \cap R_1$;

(b) either $|f_{i1}(z)| \leq |f_{i2}(z)|$ for all $z \in F_i$ or $|f_{i1}(z)| > |f_{i2}(z)|$ for all but finitely many $z \in F_i \cap R_1$.

Using Corollary 3.10(i) and Theorem 3.9, we see that the desired result holds for τ and these $F_1, \ldots, F_k$. □

Lemma 4.2 *Let F be a connected R_1-affinoid, $f \in \mathcal{O}(F)$, and let $r > 1$. Then $\{z \in F \cap R_1 : K_1 \models P_r(f(z))\}$ is semialgebraic.*

Proof. Let $f(z) = (\psi^* E)(z).r(z)$ as given by Corollary 3.4. Since $E(Y)$ is a unit in $R_1\{Y\}$, there is a positive integer M such that for all $y \in R_1^n$ the coset of $E(y)$ modulo the multiplicative group of nonzero rth powers of K_1 depends only on the congruence class $E(y) + p^M R_1$, which depends only on the tuple $y + p^M R_1^n$. Hence there is a partition of $F \cap R_1$ into finitely many semialgebraic subsets $A_1, \ldots, A_s$, such that for each $i \in \{1, \ldots, s\}$ there is nonzero $\lambda_i \in K_1$ such that $P_r\big(\lambda_i \psi^*(E)(z)\big)$ holds for all $z \in A_i$. Thus, the required set is

$$\bigcup_{i=1}^{s} \big\{ z \in A_i : K_1 \models P_r(\lambda_i^{-1} r(z)) \big\},$$

and this is clearly semialgebraic. $\square$

Proof of Theorem A′. Although Theorem A′ is stated for all $\mathcal{L}_{an}$-elementary extensions of $\mathbf{Q}_p$, it suffices to prove it for just a single κ^+-saturated $\mathcal{L}_{an}$-elementary extension, where κ is the cardinality of $\mathbf{Q}_p$. Hence there is no loss of generality in proving it just in the setting of this section.

It suffices to show that if $\psi(z)$ is an $\mathcal{L}_{an}^D$-formula in one variable with parameters from R_1, then ψ defines a semialgebraic subset of R_1. By the quantifier-elimination theorem of Denef and van den Dries [2], we may reduce to the case that ψ is atomic, so is one of the forms $\tau(Z) = 0$ or $P_n\big(\tau(Z)\big)$, where $\tau(Z)$ is an $\mathcal{L}_{an}^D$-term. By Proposition 4.1 and Lemma 3.1, it suffices to show that if F is a connected R_1-affinoid and $\tau, \tau_1, \tau_2 \in \mathcal{O}(F)$, then the sets $\{z \in F : \tau(z) = 0\}$, $\{z \in F : |\tau_1(z)| \leq |\tau_2(z)|\}$ and $\{z \in F : P_n\big(\tau(z)\big)\}$ are semialgebraic. This however is immediate from Theorem 3.9(i), Corollary 3.10(ii) and Lemma 4.2. $\square$

5 One-dimensional subanalytic sets are semianalytic

In this section we extend the result 3.31 from [2] that 1-dimensional subanalytic subsets of $\mathbf{Z}_p^n$ are semianalytic. A minor difficulty is that the definition of 'semianalytic set' is local and therefore inappropriate for elementary extensions of the standard model. The solution of this difficulty involves fractional linear transformations, which live on the projective line rather than on the field.

Let K_1 be an elementary extension of the $\mathcal{L}_{an}$-structure $\mathbf{Q}_p$, with valuation ring R_1. We make the projective line

$$\mathbf{P}(K_1) := K_1 \cup \{\infty\}$$

into a topological space so that K_1 equipped with the valuation topology is an open subspace and the sets $\{z \in K_1 : |z| > |a|\} \cup \{\infty\}$ with $a \in K_1$ form a basis of neighborhoods of ∞. Each power $\mathbf{P}(K_1)^n$ is equipped with the product topology. The p-adic projective line $\mathbf{P}(\mathbf{Q}_p)$ and its powers are p-adic analytic manifolds, for which therefore the notions of semianalytic and subanalytic subset are defined. Given $a, b, c, d \in K_1$ with $ad - bc \neq 0$ the fractional linear transformation $z \mapsto (az+b)/(cz+d)$ is a homeomorphism from $\mathbf{P}(K_1)$ onto itself (an analytic isomorphism if $K_1 = \mathbf{Q}_p$).

We define the dimension $\dim(S)$ of a nonempty definable set $S \subseteq R_1^n$ as the largest natural number d for which there is a sequence $i = (i_1, ..., i_d)$ with $1 \leq i_1 < \ldots < i_d \leq n$ such that $\pi_i(S)$ has nonempty interior in R_1^d, where $\pi_i : R_1^n \to R_1^d$ is the projection map given by $\pi_i(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_d})$; we also put $\dim(\emptyset) = -\infty$. This definition is consistent with that in [2] for subanalytic subsets of $\mathbf{Z}_p^n$. Various elementary properties of this dimension function such as its additivity under cartesian products and invariance under definable bijections are easily established, using [2] if necessary.

Definition 5.1 *A* special map *is a map* $\mathbf{P}(K_1)^n \to \mathbf{P}(K_1)^{n+k}$ *of the form* $x \mapsto (x, \alpha_1(x_{i_1}), \ldots, \alpha_k(x_{i_k}))$ *for some* k *and some* $i_1, \ldots, i_k \in \{1, \ldots, n\}$ *such that each* $\alpha_j : \mathbf{P}(K_1) \to \mathbf{P}(K_1)$ *is a fractional linear transformation.*

Note that special maps are injective and for $K_1 = \mathbf{Q}_p$ are also analytic.

Theorem B *Each definable set* $S \subseteq R_1^n$ *of dimension at most* 1 *is of the form* $S = R_1^n \cap \mu^{-1}(T)$ *for some special map* $\mu : \mathbf{P}(K_1)^n \to \mathbf{P}(K_1)^{n+k}$ *and some set* $T \subseteq R_1^{n+k}$ *that is quantifier-free definable in the* $\mathcal{L}_{an}$*-structure* R_1.

This theorem implies the result from [2] that subanalytic subsets of $\mathbf{Z}_p^n$ of dimension at most 1 are semianalytic. To see why, note that subsets of $\mathbf{Z}_p^n$ that are quantifier-free definable in the language $\mathcal{L}_{an}$ are semianalytic, and that inverse images of semianalytic sets under analytic maps between p-adic analytic manifolds are again semianalytic.

In the proof of the theorem we will tacitly use the following easy observation: if the theorem holds for definable sets $S_1, S_2 \subseteq R_1^n$, then it holds for their union $S = S_1 \cup S_2$. It is convenient here to work with a valuation v rather than norms. The value group of a valued field L will be denoted vL.

Lemma 5.2 *Let* r *be a positive integer,* $P_r(K_1)$ *the set of nonzero* r*th powers in* K_1. *There is a semialgebraic function* $f_r : P_r(K_1) \cup \{0\} \to K_1$ *such that* $(f_r(x))^r = x$ *for all* $x \in P_r(K_1) \cup \{0\}$.

Proof. This is true for the standard model $\mathbf{Q}_p$, since we have Skolem functions for finite sets in the language of rings with P_r predicates [1, Lemma 7.1]. The formula which defines f_r there will define a function with the same properties in K_1. □

From now on, we will denote $f_r(x)$ by $x^{1/r}$.

Lemma 5.3 *Let* S *be an* $\mathcal{L}_{\mathrm{an}}^D$*-substructure of* R_1, $L \subseteq K_1$ *its field of fractions. Let* $L[t^{1/r}]$ *be an extension of* L *in* K_1, *where* r *is prime, and* $t \in S$ *is such that* $t \in P_r(K_1)$ *and* $(1/r)v(t) \notin vL$. *Let* $S' = R_1 \cap L[t^{1/r}]$ *be the valuation ring of* $L[t^{1/r}]$. *Then* S' *is an* $\mathcal{L}_{\mathrm{an}}^D$*-substructure of* R_1.

Proof. It suffices to show that for any $F(Y) \in R\langle Y\rangle$ and for any $\alpha \in S'^n$, we have $F(\alpha) \in S'$. We introduce new variables A_{ij}, for $1 \le i \le n$, $0 \le j \le r-1$, and put $\widetilde{Y}_i = A_{i0} + \cdots + A_{ir-1}$ and $A = (A_{10}, \ldots, A_{1,r-1}, \ldots, A_{n0}, \ldots, A_{n,r-1})$. It is clear that we can obtain an expansion

$$F(\widetilde{Y}_1, \ldots, \widetilde{Y}_n) = \sum G_k(\widetilde{U})A^k \ ,$$

where the summation is over the tuples $k = (k_{10}, \ldots, k_{1,r-1}, \ldots, k_{n0}, \ldots, k_{n,r-1})$ with $0 \le k_{ij} \le r-1$ for all i,j, and where each $G_k(U) \in R\langle U\rangle$, with $U = (U_{ij} : 1 \le i \le n, 0 \le j \le r-1)$ a tuple of new indeterminates and $\widetilde{U} = (A^r_{ij} : 1 \le i \le n, 0 \le j \le r-1)$. Let now $\alpha = (\alpha_1, \ldots, \alpha_n) \in S'^n$ and write each $\alpha_i = \sum_{j=0}^{r-1} \alpha_{ij}(t^{1/r})^j$, where all α_{ij} are in L. Since $v(\alpha_i) \ge 0$ and $(1/r)v(t) \notin vL$, it follows as r is prime that $v(\alpha_{ij}(t^{1/r})^j) \ge 0$ for $0 \le j \le r-1$, and in particular that $\alpha^r_{ij}t^j \in S$ for all i,j. Hence the substitution of $\alpha_{ij}(t^{1/r})^j$ for A_{ij} in the expansion above gives

$$F(\alpha) = \sum G_k(\widetilde{u})a^k \in S',$$

where

$$\widetilde{u} = (\alpha^r_{ij}t^j : 1 \le i \le n, 0 \le j \le r-1)$$

and

$$a = (\alpha_{ij}(t^{1/r})^j : 1 \le i \le n, 0 \le j \le r-1).$$

□

Corollary 5.4 *Let S, L be as in Lemma 5.3, L_1 the p-adic closure of L in K_1 and $S_1 = R_1 \cap L_1$ its valuation ring. Then S_1 and L_1 are $\mathcal{L}_{an}$-elementary substructures of R_1 and K_1 respectively.*

Proof. First we note that the valuation ring S is henselian. For this, it suffices that for any $a_1, \ldots, a_n \in S$, the polynomial $q(T) = 1 + T + pa_1T^2 + \cdots + pa_nT^{n+1}$ has a zero in S. Consider the polynomial $Q(Y,T) = 1 + T + pY_1T^2 + \cdots + pY_nT^{n+1} \in \mathbf{Z}_p\langle Y\rangle[T]$. Since $Q(Y,-1) \in p\mathbf{Z}_p\langle Y\rangle$ and $\frac{\partial Q}{\partial T}(Y,-1)$ is a unit in $\mathbf{Z}_p\langle Y\rangle$, it follows from Hensel's lemma for the (p-adically complete) ring $\mathbf{Z}_p\langle Y\rangle$ that there is $g(Y) \in \mathbf{Z}_p\langle Y\rangle$ with $g(0) = -1$ and $Q(Y, g(Y)) = 0$. The substitution of $a = (a_1, \ldots, a_n)$ for Y gives $q(g(a)) = 0$. It follows from Section 3.2 of [8] that L_1 is the union of a continuous well-ordered chain $(M_\mu : \mu < \kappa)$ of subfields of K_1, with $M_0 = L$, and $M_{\mu+1} = M_\mu[t_\mu^{1/r_\mu}]$ as in Lemma 5.3. This lemma now implies that S_1 and L_1 are $\mathcal{L}^D_{an}$-substructures of R_1 and K_1 respectively. Hence, S_1 and L_1 are $\mathcal{L}_{an}$-elementary submodels of R_1 and K_1 respectively by Proposition 2.3 of [3]. □

The next result is proved in the same setting as section 4, so besides K_1 we also have $\widehat{K}_1$ around to make sense of holomorphic functions on connected

R_1-affinoids. In particular, the theorem applies to the standard model $(\mathbf{C}_p, \mathbf{Q}_p)$.

Theorem 5.5 *Let the function $\alpha : R_1 \to K_1$ be $\mathcal{L}_{an}$-definable. Then there is a partition $\mathcal{P}$ of R_1 into finitely many semialgebraic sets A, and with each $A \in \mathcal{P}$ there are associated a connected R_1-affinoid $F \supseteq A$, positive integers $r_1, \ldots, r_k$ and functions $a_i^A, b^A, f_j^A \in \mathcal{O}(F)$, where $i = (i_1, \ldots, i_k)$ with $0 \le i_1 < r_1, \ldots, 0 \le i_k < r_k$ and $j = 1, \ldots, k$ such that for all $z \in A$ we have*

$$f_1^A(z) \in P_{r_1}(K_1), \ldots, f_k^A(z) \in P_{r_k}(K_1), \quad b^A(z) \neq 0 \quad \textit{and}$$

$$\alpha(z) = \sum_i \frac{a_i^A(z)}{b^A(z)} \big(f_1^A(z)^{1/r_1}\big)^{i_1} \cdots \big(f_k^A(z)^{1/r_k}\big)^{i_k}.$$

Proof. Let $(\widehat{K}_2, K_2)$ be a κ^+-saturated elementary extension of $(\widehat{K}_1, K_1)$, where $\kappa = \mathrm{card}(K_1)$, and let $\widehat{R}_2$ and R_2 be the valuation rings of $\widehat{K}_2$ and K_2, respectively. We denote the canonical extension of α to a function $R_2 \to K_2$ also by α. Fix $z \in R_2$, and put

$$S := \{\tau(z) : \tau(Z) \text{ is an } \mathcal{L}_{an}^D\text{-term with parameters from } R_1\},$$

the $\mathcal{L}_{an}^D$-substructure of R_2 generated by z over R_1. Let L be the fraction field of S in K_2 and L_2 the p-adic closure of L in K_2. Then L_2 is an $\mathcal{L}_{an}$-elementary submodel of K_2 by Corollary 5.4. Hence $\alpha(z) \in L_2$, and thus (see section 3.2 of [8]), $\alpha(z) \in L[t_1^{1/r_1}, \ldots, t_k^{1/r_k}]$ for certain positive integers $r_1, \ldots, r_k$ and elements $t_j \in P_{r_j}(L_2) \cap S$. That is

$$\alpha(z) = \sum_i \frac{a_i(z)}{b(z)} (f_1(z)^{1/r_1})^{i_1} \cdots (f_k(z)^{1/r_k})^{i_k},$$

where the summation is over all tuples $i = (i_1, \ldots, i_k)$ with $0 \le i_1 < r_1$, …, $0 \le i_k < r_k$ and the a_i, b and $f_1, \ldots, f_k$ are 1-variable $\mathcal{L}_{an}^D$-terms with parameters from R_1, $b(z) \neq 0$. Here $r_1, \ldots, r_k$ and these terms depend on z, but as K_2 is κ^+-saturated, only finitely many possibilities for $(r_1, \ldots, r_k; (a_i), b; f_1, \ldots, f_k)$ are actually needed as z ranges through R_2, in particular as z ranges through R_1. We now apply Proposition 4.1 and Theorem A$'$. □

Let $S \subseteq R_1^{n+1}$ be a one-dimensional $\mathcal{L}_{an}$-definable set, and let $\pi_1 : R_1^{n+1} \to R_1$ be the projection onto the first coordinate. For each $x \in \pi_1(S)$, let $S_x := \{y \in R_1^n : (x, y) \in S\}$. The next lemma follows from 3.22 and 3.24 of [2].

Lemma 5.6 *There is $N \in \mathbf{N}$ such that for all $x \in \pi_1(S)$, either $|S_x| \le N$ or S_x is infinite. The latter case can only happen for finitely many x.*

We also have, from Theorem 3.6 of [2], that R_1 has definable Skolem functions in $\mathcal{L}_{an}$.

Proof of Theorem B.

The proof is by induction on n. For $n=1$ the result follows from Theorem A, with $k=0$. So assume it is true for a certain $n>0$, and let $S \subseteq R_1^{n+1}$ be an $\mathcal{L}_{an}$-definable set of dimension 1. We use the induction hypothesis and the observation before Lemma 5.2 to reduce to the case that for some $N \in \mathbf{N}$ we have $|S_x| \leq N$ for all $x \in R_1$. Then by definability of Skolem functions, we reduce further to the case that S is the graph of a definable map $\alpha = (\alpha_1, \ldots, \alpha_n) : A \to R_1^n$, where A is a semialgebraic subset of R_1. By Theorem 5.5, we can assume that $A \subseteq F$, where F is a connected R_1-affinoid, and we have for $\ell = 1, \ldots, n$ and all $x \in A$,

$$\alpha_\ell(x) = \sum_{i=(0,\ldots,0)}^{(N_{\ell 1}-1,\ldots,N_{\ell k}-1)} \frac{a_{\ell i}(x)}{b_\ell(x)} \left(f_{\ell 1}(x)^{1/N_{\ell 1}}\right)^{i_1} \cdots \left(f_{\ell k}(x)^{1/N_{\ell k}}\right)^{i_k},$$

for some $k \in \mathbf{N}$, positive integers $N_{\ell 1}, \ldots, N_{\ell k}$ and functions $a_{\ell i}, b_\ell, f_{\ell 1}, \ldots, f_{\ell k} \in \mathcal{O}(F)$, with $b_\ell(x) \neq 0$ and $f_{\ell j}(x) \in P_{N_{\ell j}}(K_1)$ for all $x \in A$. This expresses α_ℓ as a semialgebraic function of the $a_{\ell i}$, b_ℓ and $f_{\ell j}$. Hence we can apply Macintyre's quantifier elimination theorem [7] to obtain an equivalence, for $(x,y) \in R_1^{1+n}$:

$$(x,y) \in S \text{ if and only if } R_1 \models \theta(x, a(x), b(x), f(x))$$

for some quantifier-free formula θ in the language of rings with P_r predicates, where $a(x) := (a_{\ell i}(x))$, $b(x) := (b_\ell(x))$ and $f(x) := (f_{\ell j}(x))$. By writing all $a_{\ell i}, b_\ell, f_{\ell j} \in \mathcal{O}(F)$ in the form $g \circ \psi$ for suitable $g \in K_1\{Y\}$ and ψ as in section 3.2, we obtain the desired result. $\square$

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