Independence for types in algebraically closed valued fields

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Shelah's program (1970's)

Problem

Given a theory T and a cardinal $\lambda > |T|$, let $I(T, \lambda)$ be the number of models of T of cardinality λ, up to isomorphism.

Question

what can the function $I(T, \lambda)$ be?

Dichotomy

T unstable

$I(T, \lambda) = 2^\lambda$ (the maximum possible value)

Examples of unstable structures include any theory of a structure with an ordering or a valuation.

T stable

$I(T, \lambda) = ?$

Examples include algebraically closed fields, separably closed fields, differentially closed fields.

Dichotomy

Pure model theory worked on:

- tools to study stable theories, the most fundamental being *forking* and its attendant concepts of independence, dimension, orthogonality, canonical base
- unstable theories no longer of interest to classification theory

Many mathematically interesting theories are unstable, and these continued to be studied by applied model theorists, using various methods including quantifier elimination.

Amalgamation?

The two branches of model theory have come back together in the last 15 years, as it has become clear that some of the ideas and tools of stability theory can be used also in the unstable context.

Terminology for independence

Fix a theory T, and a model (the monster model) $\mathcal{U}$ large, sufficiently saturated. All sets of parameters will be taken from $\mathcal{U}$. Mostly, a will be a tuple from $\mathcal{U}$, and $B \supseteq C$ will be sets of parameters from $\mathcal{U}$.

To say that (the type of) a is independent from B over C

$$a \downarrow_C B$$

should say that B knows no more about a than C does.
In other words, allowing parameters from B (and hence potentially increasing the expressibility of the formulae) does not decrease the set of realisations of $\text{tp}(a/C)$.

Potential properties of an independence notion

Monotonicity if $a \downarrow_C B$ and $C' \subseteq B' \subseteq B$ then $a \downarrow_{C'} B'$

Transitivity $a \downarrow_C B$ and $a \downarrow_B B'$ then $a \downarrow_C B'$

Symmetry if $a \downarrow_C B$ and $b \in B$ then $b \downarrow_C a \cup C$

Finite character $a \downarrow_C B$ if and only if $a \downarrow_C C \cup B_0$ for every finite $B_0 \subset B$

Local character $a \downarrow_C B$ if for every $C_0 \subset C$ of cardinality at most T, $a \downarrow_{C_0} B$

Independence theorem

Existence of invariant extensions

Uniqueness of invariant extensions

Simplicity versus stability

Theorem (B. Kim and A. Pillay)

1. *If $\downarrow$ satisfies all of the above properties then it is the independence derived from non-forking and the theory is stable.*
2. *If $\downarrow$ satisfies all but the last of the above properties then it is the independence derived from dividing and the theory is simple.*

Valued fields

Definition

A valuation is a map $v : K \to \Gamma \cup \{\infty\}$ from a field K to an ordered group Γ such that for all $a, b \in K$

$$\begin{aligned} v(ab) &= v(a) + v(b) \\ v(a+b) &\geq \min\{v(a), v(b)\} \\ v(0) &= \infty \end{aligned}$$

valuation ring $\mathcal{O}_K = \{x \in K : v(x) \geq 0\}$
maximal ideal $\mathcal{M} = \{x \in K : v(x) > 0\}$
residue field $\mathcal{O}_K/\mathcal{M} = k$

Examples of valued fields

A good class of examples is given by fields of formal Laurent series. Given a field F, define

$$F((t)) = \{ \sum_{i=-M}^{\infty} q_i t^i : q_i \in F, q_{-M} \neq 0, M \in \mathbb{Z}\}$$

Then

$$\begin{aligned} v : F((t)) &\rightarrow \mathbb{Z} \\ \text{defined by} \quad v(\sum_{i=-M}^{\infty} q_i t^i) &= -M \end{aligned}$$

is a valuation. In this case, $\Gamma = \mathbb{Z}$ and $k = F$.

Examples of valued fields

If F is algebraically closed then the field of Puiseux series

$$F\langle\langle t\rangle\rangle = \bigcup_{n=1}^{\infty} F((t^{\frac{1}{n}}))$$

is algebraically closed. Elements are power series as above in fractional powers of t, the valuation is defined the same way. Now $\Gamma = \mathbb{Q}$ and $k = F$.

Examples of algebraically closed valued fields

$\mathbb{C}\langle\langle t\rangle\rangle$ is an algebraically closed valued field of characteristic $(0,0)$

$\tilde{\mathbb{F}}_p\langle\langle t\rangle\rangle$ is an algebraically closed valued field of characteristic (p,p)

$\tilde{\mathbb{F}}_p\langle\langle p\rangle\rangle$ is an algebraically closed valued field of characteristic $(0,p)$, where here the definition of addition and multiplication in the field are modified to adapt to the fact that p is a prime, and not an indeterminate.

ACVF

It is straightforward to write down the axioms for a valued field, say in a language of fields with a predicate for the relation $v(x) \geq v(y)$. Add the axiom scheme to enforce that the field is algebraically closed. We call this theory ACVF.

To get a complete theory one only has to specify the field and residue field characteristic.

Theorem (A. Robinson, mid 1950's)

ACVF has quantifier elimination.

Fundamental properties of definable sets in ACVF

Corollary

Let K be an ACVF. Definable sets are boolean combinations of sets of the form

$$\{x : f(x) = 0\}$$

$$\{x : v(f(x)) \geq v(g(x))\}$$

where f, g are polynomials over K.

In particular, a definable set in one variable is either finite or has non-empty interior in the valuation topology.

In fact, more is true. Since K is algebraically closed, the polynomials can be factored into linear terms. Hence an infinite definable set in one variable is a boolean combination of balls.

$\{x \in K : v(x-c) \geq \gamma\}$ is the *closed ball* of radius γ around c

$\{x \in K : v(x-c) > \gamma\}$ is the *open ball* of radius γ around c.

Ax-Kochen, Ersov, about 1965

Theorem (J. Ax and S. Kochen, Y. Ersov)

Let K, L be henselian valued fields of characteristic 0. Then $K \equiv L$ if and only if

- *the value groups are elementarily equivalent and*
- *the residue fields are elementarily equivalent*

A good theory to work with

Thus ACVF seems a prime candidate for having a good notion of independence:

- the residue field is algebraically closed (and stably embedded), hence strongly minimal
- the value group is divisible (and stably embedded), hence o-minimal
- the full theory seems to be controlled in some way by the theories of its residue field and value group.

A first approach

Work by analogy with equivalent definitions of independence in an algebraically closed field.

References

The results about algebraically closed valued fields from here on can be found in

- D. Haskell, E. Hrushovski and D. Macpherson,“Stable domination and independence for algebraically closed valued fields”, preprint, www.math.mcmaster.ca/ haskell/index.html
- D. Haskell and D. Macpherson, “Definable sets in valued fields”, *Proceedings of the Euro-Conference on Model Theory and Applications, Ravello,* (2002).

Independence in an algebraically closed field

Morley Rank

$a \downarrow_C B \iff \mathrm{RM}(a/B) = \mathrm{RM}(a/C)$

Genericity

$a \downarrow_C B \iff$ whenever a is generic over C in a variety V defined over C, then a remains generic in V over B

Polynomial ideals

$a \downarrow_C B \iff I(a/B) = I(a/C)$, where $I(a/B)$ is the ideal of polynomials with coefficients in B vanishing at a

Morley Rank

Let D be a definable set in some ($\aleph_0$-saturated) structure.

Definition

- $\mathrm{RM}(D) \geq 0$ iff $D \neq \emptyset$
- $\mathrm{RM}(D) \geq \alpha + 1$ iff there is an infinite family D_i of pairwise disjoint definable sets contained in D with $\mathrm{RM}(D_i) \geq \alpha$ for all i
- for α a limit ordinal, $\mathrm{RM}(D) \geq \alpha$ iff $\mathrm{RM}(D) \geq \beta$ for every $\beta < \alpha$

If $D = \emptyset$ then $\mathrm{RM}(D) = -1$. If $\mathrm{RM}(D) \geq \alpha$ and $\mathrm{RM}(D) \not\geq \alpha + 1$ then $\mathrm{RM}(D) = \alpha$. If $\mathrm{RM}(D) \geq \alpha$ for all ordinals α then $\mathrm{RM}(D) = \infty$.

$\mathrm{RM}(\mathrm{tp}(a/C)) = \inf\{\mathrm{RM}(D) : D \text{ is a } C\text{-definable set containing } a\}$

Morley Rank

Fact

A theory T is ω-stable if and only if every type has a Morley rank.

In this case, $\downarrow$ as defined by Morley rank has all the stated properties.

Thus we do not expect to be able to use Morley rank to define independence for all types in an ACVF. But perhaps we can use it for *some* types?

No.

Proposition

Let K be an ACVF, D an infinite definable subset of K^1. Then $\mathrm{RM}(D) = \infty$.

Morley Rank in ACVF

Proof. Suppose $\mathrm{RM}(D) = \alpha$. Then by definition, D does not contain an infinite disjoint family of subsets D_i of rank α.

Since D is infinite, it contains a closed or open ball; without loss of generality, we may assume that $D = \{x \in K : v(x - c) > \gamma\}$.

Let $\gamma < \gamma_1 < \gamma_2 < \cdots$ be an infinite increasing sequence from Γ. Let $c_i \in D$ be such that $v(c - c_i) = \gamma_i$, and let $D_i = \{x \in D : v(x - c_i) > \gamma_i\}$.

By the superadditivity of the valuation, for $i < j$,

$$v(c_i - c_j) \geq \min\{v(c_i - c), v(c - c_j)\} = \gamma_i.$$

Thus $D_i \cap D_j = \emptyset$; since, if $v(x - c_i) > \gamma_i$, then

$$v(x - c_j) \geq \min\{v(x - c_i), v(c_i - c_j)\} = \gamma_i < \gamma_j.$$

But D_i is isomorphic to D, by the map $x \longrightarrow \frac{d(x-c_i)}{d_i} + c$, where $v(d) = \gamma$, $v(d_i) = \gamma_i$.
Thus $\mathrm{RM}(D_i) = \mathrm{RM}(D)$.

Contradiction.

Genericity in ACF

Definition

a is generic in an irreducible algebraic variety V defined over C if $a \in V$ and for any other irreducible algebraic variety W defined over C, if $a \in W$ then $V \subseteq W$.

If a is generic in V over C then $\mathrm{RM}(a/C) = \mathrm{RM}(V)$.

$$\text{Thus } a \downfree_C B \iff \mathrm{RM}(a/B) = \mathrm{RM}(V)$$
$$\iff a \text{ remains generic in } V \text{ over } B$$

Varieties are sets defined by positive quantifier-free formulae which, by quantifier elimination for ACF, are basic sets for all definable sets.

Genericity in ACVF

Positive quantifier-free formulae involve the valuation, and by quantifier elimination for ACVF, the basic sets for all the definable sets are the balls. More generally, we need to refer to *unary sets*.

Definition

a is generic over C in a unary set V defined over C if $a \in V$ and for any other C-definable unary set W defined over C, if $a \in W$ then $V \subseteq W$.

Definition

Suppose a is generic in V over C. Then $a \downarrow^g_C B$ if a is still generic in V over B.

Examples of generic independence

a generic in $\mathcal{O}_K$ over $C = \mathrm{acl}(\emptyset)$

Then $v(a) \geq 0$, but $a \notin \mathcal{M}$, so $v(a) = 0$.
Also, a is not in any open C-definable ball with radius 0, so the residue of a is not in the residue field of C.

Let a' be another generic element of $\mathcal{O}_K$ over a; $a' \downarrow^g_C a$.
Then $v(a') = 0$, and also $v(a - a') = 0$.
Thus $a \downarrow^g_C a'$.

Examples of generic independence

a generic in $\mathcal{M}$ over $C = \mathrm{acl}(\emptyset)$

Then $v(a) > 0$, but if $\gamma \in \Gamma(C)$ with $\gamma > 1$, a is not in the ball around 0 of radius γ, so $v(a) < \gamma$. Thus $v(a)$ is not in $\Gamma(C)$.

Let a' be another generic element of $\mathcal{M}$ over a; $a' \mathop{\smile\!\!\!\!\!\mid}^{g}_{C} a$.

Then $0 < v(a') < v(a)$.

Thus $a \not\mathop{\smile\!\!\!\!\!\mid}^{g}_{C} a'$.

Sequential independence

Definition

Let $a = (a_1, \ldots, a_n)$, $U = (U_1, \ldots, U_n)$. We say that $a \mathop{\smile\!\!\!\!\!\!\!\;|}^{g}_{C} B$ via U if either $a \in \mathrm{acl}(C)$ or for each i, U_i is an $\mathrm{acl}(Ca_1 \ldots a_{i-1})$-definable unary set and a_i is generic in U_i over $Ba_1 \ldots a_{i-1}$.

Properties of sequential independence

- transitivity, monotonicity
- symmetry fails
- finite character
- existence and uniqueness of invariant extensions

Independence via an ideal of polynomials: ACF

Definition

$$\begin{aligned} I(a/C) &= \{f(X) \in K[X] : \mathrm{tp}(a/C) \vdash f(x) = 0\} \\ I(V) &= \{f(X) \in K[X] : \forall x \in V\, f(x) = 0\} \end{aligned}$$

For a generic in V over C, $I(a/C) = I(V)$.

Thus $a \mathop{\smile\!\!\!\!\!\!\mid}_C B \iff I(a/C) = I(a/B)$.

Independence via a module of polynomials: ACVF

Definition

$$J(a/C) = \{f(X) \in K[X] : \mathrm{tp}(a/C) \vdash v(f(x)) \geq 0\}$$

$$a \mathop{\smile\!\!\!\!\!\!|}^{J}_{C} B \iff J(a/C) = J(a/B)$$

Example where $\mathop{\smile\!\!\!\!\!\!\!\;|}^{J}$ fails

As in second example previously, take a generic in $\mathcal{M}$, a' generic over a in $\mathcal{M}$.

$$a' \mathop{\smile\!\!\!\!\!\!\!\;|}^{g} a \quad , \qquad a \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{g} a', \qquad \text{and } 0 < v(a') < v(a)$$

Let $f(X) = \frac{1}{a'}(X - a')$. Then $\mathrm{tp}(a/a') \vdash v(f(x)) = 0$, hence $f(X) \in J(a/a')$.
But $\mathrm{tp}(a/\emptyset)$ allows $v(f(x)) < 0$, so $f(X) \notin J(a/\emptyset)$. Thus

$$a \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{J} a' \; .$$

Equivalence for 1-types

Proposition

Let $a \in K^1$. Then

$$a \mathop{\smile\!\!\!\!\!\!\downarrow}^{J}_{C} B \iff a \mathop{\smile\!\!\!\!\!\!\downarrow}^{g}_{C} B.$$

However, this is not true in general.

Properties of J-independence

- monotonicity, transitivity
- symmetry fails
- finite character

Another approach

Instead of working by analogy, work with the strongly minimal structure that is already there.

The residue field k is stable and stably embedded. For a parameter set C, define the stable reduct of C to be
$St_C =$ the multisorted structure whose sorts are the C-definable stable, stably embedded subsets of $\mathcal{U}$ and whose relations are all the C-definable relations on these sorts
For $A \subset \mathcal{U}$, $St_C(A) = St_C \cap \mathrm{dcl}(A)$. If C is a model of the theory, then St_C is just k^{eq} with a family of k-vector spaces.
But if C is not a model, then there can be stable, stably embedded sets defined over C which cannot be identified with sets definable over the residue field.

Stable domination

Definition

$A \downarrow_C^{dom} B$ if $St_C(A) \downarrow_C St_C(B)$ and $\mathrm{tp}(St_C(A)/CB) \vdash \mathrm{tp}(A/CB)$.

Definition

Also $\mathrm{tp}(A/C)$ is *stably dominated* if whenever $St_C(A) \downarrow_C St_C(B)$, $A \downarrow_C^{dom} B$.

Properties of domination independence

symmetry, monotonicity

finite character, local character

existence and uniqueness of invariant extensions

Orthogonality to the value group

Definition

Assume C is algebraically closed. We say that $\mathrm{tp}(a/C)$ is *orthogonal* to Γ if for any model $M \supseteq C$, if $a \downarrow^{g}_{C} M$ then $\Gamma(Ma) = \Gamma(M)$.

Theorem

$\mathrm{tp}(a/C)$ *is stably dominated if and only if* $\mathrm{tp}(a/C)$ *is orthogonal to* Γ.

Theorem

If $\mathrm{tp}(a/C)$ *is orthogonal to* Γ *then all notions of independence are equivalent.*

Within ACVF

- Understand better the axiomatic properties of all the notions of independence.
- In particular, J-independence is quite hard to work with.
- Do J-independent extensions exist?

Expansions of ACVF

ACVF with restricted analytic functions

Adjoin to the language function symbols for functions defined by convergent power series. The theory has quantifier elimination (in a slightly larger language) (Lipshitz, 1993).

Questions

Does it eliminate imaginaires to the same sorts as the algebraic structure? What do the independence theories look like in this case? (work in progress with David Lippel)

Expansions of ACVF

ACVF with a derivation or an automorphism

Adjoin to the language a function symbol for a function which is a derivation or an automorphism. The theory of the differential or difference closure has quantifier elimination (Scanlon, 2001).

Question

Does it eliminate imaginaries to the same sorts as the algebraic structure? What do the independence theories look like? (work in progress with Tom Scanlon and Yoav Yaffe)

Expansions of ACVF

V-minimal structures

A concept recently introduced by Hrushovski. A v-minimal structure looks like a valued field, being controlled by an o-minimal part and a part which is either finite or strongly minimal. The two parts are orthogonal to each other. The v function is not exactly a valuation.

Stable domination

Applications

Structure theory for definable groups with a stably dominated type (Hrushovski). Such groups can arise in an ACVF.

In other contexts

For example, in a differentially closed valued field.

Generalisations

Very recently, Hrushovksi, Peterzil and Pillay have announced results about groups definable in o-minimal structures, using the concept of compactly dominated types.

Other independence ideas

Dependent theories

Recently, Shelah has been looking at the notion of "dependent theories", which include all the "relation-minimal" theories (o-minimal, P-minimal, C-minimal). There seems to be a notion of independence; how does it compare with the ones we have looked at here?

thorn forking

The notion of thorn forking has recently been introduced by Ealy, Onshuus and Scanlon. It works very well in o-minimal theories, but rather less well in ACVF. How does the independence derived from thorn forking compare with the ones we have seen today?

References

- D. Haskell, E. Hrushovksi and D. Macpherson, "Stable domination and independence for algebraically closed valued fields", preprint, www.math.mcmaster.ca/haskell/index.html
- D. Haskell and D. Macpherson, "Definable sets in valued fields", *Proceedings of the Euro-Conference on Model Theory and Applications, Ravello,* (2002).