Lecture 1

Chapter 2: Probability

2.1 Sample Space and Event

Sample. Random Experiment

Experiment that can result in different outcomes, even though it is repeated in the same manner.

Examples: Draw a dice

Sample Spaces

The set of all possible outcomes that an experiment. S

Set

A collection of distinct objects.

1. Distinct ~~{1, 1, 2}~~ $\{1, 2\}$
2. No Order $\{1, 2\} \Longleftrightarrow \{2, 1\}$
3. Certain ~~{All beauties}~~

$S = \{1, 2, 3, 4, 5, 6\}$

Empty set ϕ

Tree Diagram

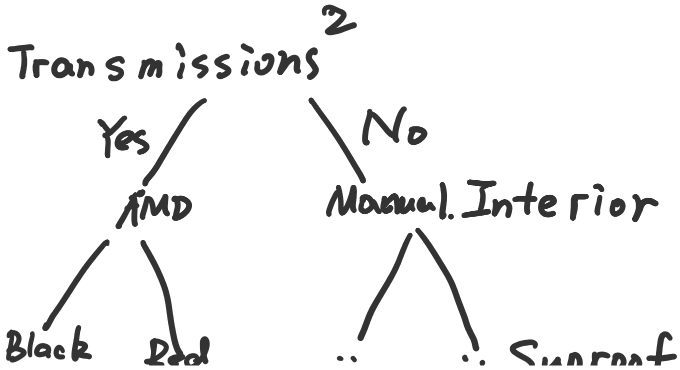
Example: Car Options:

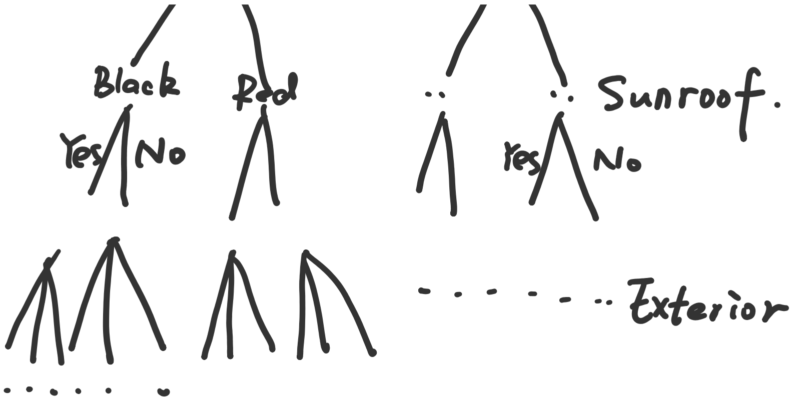
Exterior {white, black, Grey} 3

Interior {Black Red} 2

Sunroof {Yes, No} 2

Transmissions {AMD, Manual}. 2





$24 = 2 \times 2 \times 2 \times 3$

Russell's paradox. Barbers paradox.

A barber: I only shave those people who do not shave themselves.

Events:

Subset of a $\overbrace{\text{sample Space.}}^{S}$

A, B, E_1, E_2 .....

Example: $A = \{1, 2\}$ $S = \{1, 2, 3, 4, 5, 6\}$

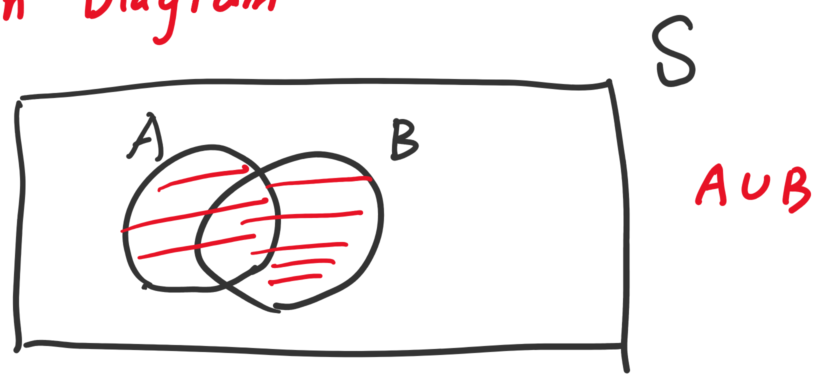
$\therefore \underbrace{P}(A) = \frac{2}{6} = \frac{\#\text{ elements of } A}{\#\text{ elements of } S}$

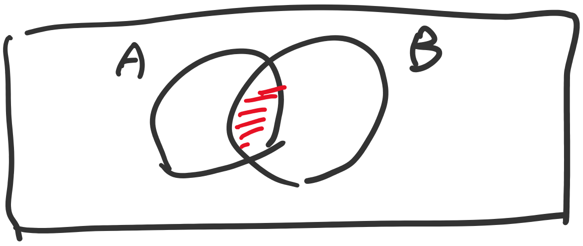
Basic Operations of Events:

A, B are two Events:

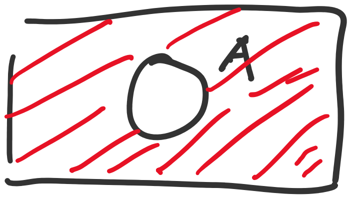
1. Union (or) : $A \cup B$
2. Intersection (&) : $A \cap B$
3. Complement (not) : A', B'

Venn Diagram





$A \cap B$



A'

Example:

$A = \{1, 2\}$
$B = \{2, 3, 4, 5\}$
$S = \{1, 2, 3, 4, 5, 6\}$

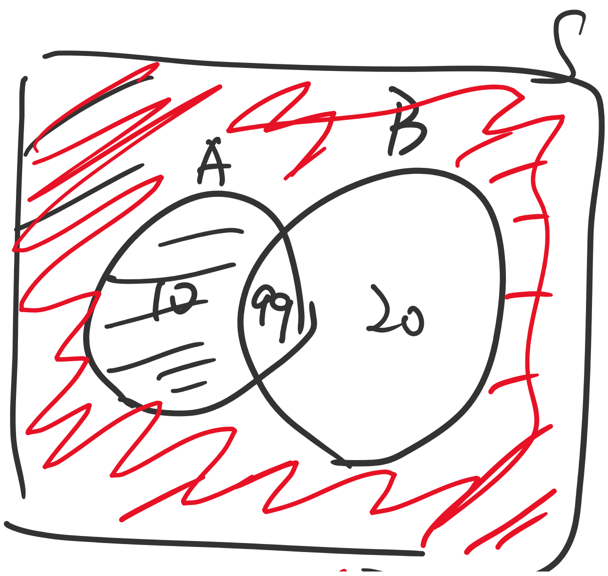
$A \cup B = \{1, 2, 3, 4, 5\}$
$A \cap B = \{2\}$
$A' = \{3, 4, 5, 6\}$
$B' = \{1, 6\}$

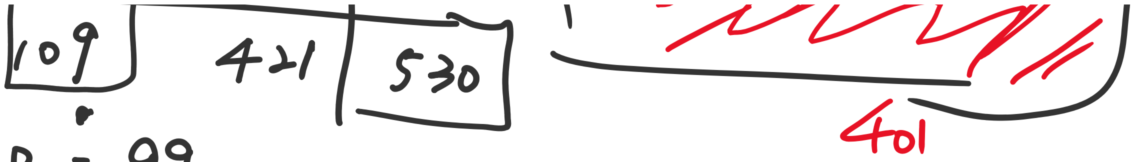
Example:

A = Patient has disease.
A' = Patient does not have disease.

B = Test Positive
B' = Test negative.

	A	A'	
B	99	20	119
B'	10	401	411
	109	421	530





$\# \; A \cap B = 99.$

$A \cup B = 99 + 20 + 10$

$= 119 + 109 - 99$

Counting Technologies.

Multiplication Rule:

k steps, $n_1, n_2, n_3 \cdots n_k$ choices.

Then, the total # of choices

$\underline{n_1 \times n_2 \cdots\cdots n_k}$

Example: web design.

four clours, Three fonts, three positions

to insert image.

$4 \times 3 \times 3 = 36.$

Factorials: (order)

Example: 4 people, 4 seats

$$\square \quad \square \quad \square \quad \square$$
$$4 \quad 3 \quad 2 \quad 1$$

$= 4 \times 3 \times 2 \times 1$

$= 4!$

n people, order them

$n! = n \times (n-1) \times (n-2) \cdots\cdots \times 3 \times 2 \times 1$

Dermutation (select with order)

Permutation (select with order)

6 people, 4 seats

$$\square \quad \square \quad \square \quad \square$$
$$6 \quad\ 5 \quad\ 4 \quad\ 3.$$

$$P^{6}_{4} = \frac{6!}{(6-4)!} = \frac{6\times5\times4\times3\times\cancel{2\times1}}{\cancel{2\times1}}$$

Combinations. (no order, just choose)

Example:

choose 4 people out of 6 to take course.

$$C^{6}_{4} = \binom{6}{4} = \frac{6!}{(6-4)!\,4!}$$

(Binomial Theorem)

$$\underbrace{C^{6}_{4}}_{\text{choose 4 people}} \cdot \underbrace{4!}_{\text{order}} = \underbrace{P^{6}_{4}}_{\text{choose with order.}}$$

$$C^{n}_{r} \cdot r! = P^{n}_{r}$$

Examples:

4 Apples. 2 oranges. $4+2=6$

select 3 without restrictions.

select 3 without restrictions.

$$\binom{6}{3} = C^6_3$$

select 3, decide order to eat them

$$P^6_3$$

select 2 Apples, 1 orange.

$$\binom{4}{2} = C^4_2 = 6$$

$$C^2_1 = 2$$

$$6 \times 2 = 12. \quad \text{no order}$$

$$12 \cdot 3! \quad \text{with order.}$$

combinations for 3 fruits

$$3 \times 2 \times 1 = 3!$$

$$C^4_2 \, C^2_1 \cdot 3!$$

choose 2 Apples — choose 1 orange — order them.

Permutations for similar things.

'Hamilton' 8 Letters

$$8!$$

'Engineer' 8 Letters

'Engineer' 8 Letters

$$\frac{8!}{3!\,2!}$$ 3 es, 2 ns.

2.2 Interpretations and Axioms of Probabilities.

Probability is used to quantify the likelihood, or chance, that an outcome of a random experiment to occur.

Event A, $\boxed{p(A) = \dfrac{\#\text{ elements in } A}{\#\text{ elements in } S}}$

Example:

	A	A'	
B	99	20	119
B'	10	401	411
	109	421	530

$p(A) = \dfrac{109}{530}$

$p(A \cap B) = \dfrac{99}{530}$.

$p(A' \cap B) = \dfrac{20}{530}$

$p(A \cup B) = \dfrac{99 + 20 + 10}{530}$

Axioms of Probability

Probability is a number that is assigned

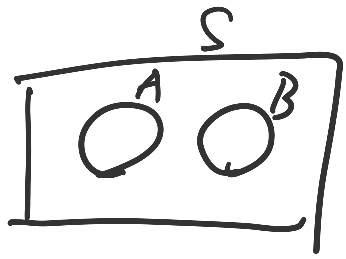
Axioms of Probability

Probability is a number that is assigned to each number of collection of events from a random experiment that

Satifies:

1. $p(S) = 1$
2. $0 \le p(E) \le 1$
3. Two events, $A \cap B = \phi$

$\therefore p(A \cup B) = p(A) + p(B)$



Mutually exclusive

$$A \cap B = \phi \iff p(A \cap B) = 0$$

more, $p(\phi) = 0$

$p(A') = 1 - p(A)$ $\qquad A \cap A' = \phi$

$\qquad A \cup A' = S$

ϕ is complement of S.

Example:

Post code questions.

L8S 4L8 McMaster.

Letters : A to N : 14 letters

digits : 1 to 8 : 8 digits.

$\#$ total : $\underbrace{14}\times\underbrace{8}\times\underbrace{14}\times\underbrace{8}\times\underbrace{14}\times\underbrace{8}$

(a) Probability that code has no repeat letters.

A = codes have no repeat letters

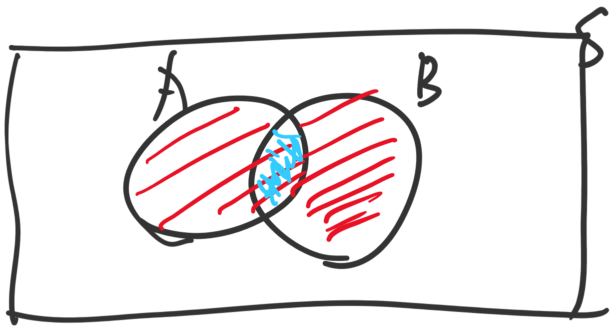
$\# A = 14\times 8\times 13\times 8\times 12\times 8$

$$p(A) = \frac{\# A}{\# \text{Total}} = \frac{\cancel{14\times 8}\times 13\times \cancel{8}\times 12\times \cancel{8}}{\cancel{14\times 8}\times 14\times \cancel{8}\times 14\times \cancel{8}}$$

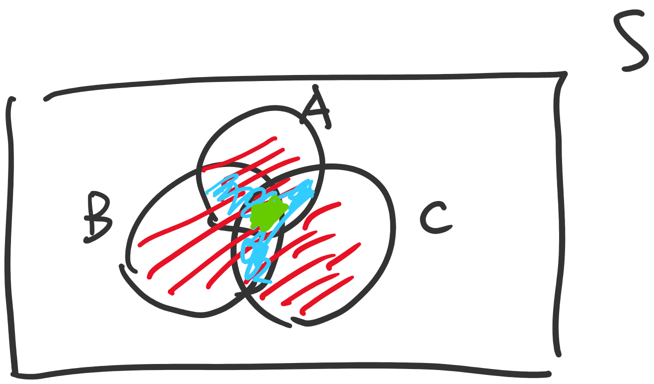
$$= \frac{13\times 12}{14\times 14} = \frac{39}{49}$$

2.3 Addition Rules.

Probability of a Union.



$$p(A\cup B) = p(A) + p(B) - p(A\cap B)$$



$$p(A\cup B\cup C) = \underline{p(A)} + \underline{p(B)} + \underline{p(C)}$$
$$- p(A\cap B) - p(A\cap C) - p(B\cap C)$$
$$+ p(A\cap B\cap C)$$

$+ P(A \cap B \cap C)$

(b) Find the probability that codes either starts with an "A" or ends with an even digit. (2, 4, 6, 8).

Events: B = The post code starts with 'A'.

C = The post code ends with an even number.

$\therefore B \cap C$

$\# B = \boxed{1} \times 8 \times 14 \times 8 \times 14 \times 8$

$\# C = 14 \times 8 \times 14 \times 8 \times 14 \boxed{\times 4}$

$\# B \cap C = \boxed{1} \times 8 \times 14 \times 8 \times 14 \times \boxed{4}$

$$P(B \cup C) = P(B) + P(C) - P(B \cap C)$$

$$= \frac{\# B + \# C - \# B \cap C}{\# \text{ total}}$$

$$= \frac{15}{28}.$$

(c). Find the probability that code starts with an 'A' and does not contain 'B'

Event D

$$\# D = \frac{1 \cdot 8 \cdot 13 \cdot 8 \cdot 13 \cdot 8}{14 \cdot 8 \cdot 14 \cdot 8 \cdot 14 \cdot 8}$$

$$\#D = \frac{\text{[illegible]}}{14 \cdot 8 \cdot 14 \cdot 8 \cdot 14 \cdot 8}$$

Example:

Fruit 38, choose 6.

16 Apples, 12 oranges, 10 bananas.

(a). The probability that all 6 selected are the same type.

$$C^{12}_{0} = 1$$

$\underset{\text{only}}{\#}$ Apples: $C^{16}_{6} \cdot C^{12}_{0} \cdot C^{10}_{0}$

$= C^{16}_{6}$

\# only oranges $= C^{12}_{6}$

\# only bananas $= C^{10}_{6}$

\# ~~total~~ $= C^{38}_{6}$

$$\therefore p(\text{same}) = \frac{C^{16}_{6} + C^{12}_{6} + C^{10}_{6}}{C^{38}_{6}}$$

(b) The probability that at least two different types are selected.

(a) and (b) are complements.

$1 - P(\text{Same})$

(c) The probability that exactly 3, 1, 2

(c) The probability that exactly 3, 1, 2 are chosen 3 types, respectively.

$\# = C^{16}_{3} \; C^{12}_{1} \; C^{10}_{2}$

3 appeles 1 orange 2 bananas.

$$p(\text{chose} \cdots) = \frac{C^{16}_{3} \; C^{12}_{1} \; C^{10}_{2}}{C^{38}_{6}}$$

2.4 Conditional Probability.

A, B

The probability of an event B under the knowledge that A has already happened.

$B|A$ $| \Leftrightarrow$ given.

$$p(B|A) = \frac{p(B \cap A)}{p(A)}$$

$A = \{2, 3, 4, 5\}$

$B = \{2, 3\}$.

$$p(B|A) = \frac{2}{4} = \frac{1}{2}$$

Example:

	A	A'	
B	99	20	119

B	[illegible]	[illegible]	[illegible]
B'	10	401	411
	109	421	530

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\#(B \cap A)}{\# A}$$

$$= \frac{99}{109}$$

The probability that a patient who indeed carrying the disease, is correctly diagnosed is $\frac{99}{109} = 90.8257\%$ (B given A).

2.5. Multiplication Rule and Total probability.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Longleftrightarrow \quad \underline{P(A \cap B) = P(B/A) \cdot P(A)}$$

$$\underline{P(A \cap B) = P(A/B) \cdot P(B)}$$

Example:

20% 3J, 80% 3Y.

60% males 40% Females 70% males 30 Females.

Both from 3J and is a male:

$20\% \cdot 60\%$

$p(A) = 20\%$

$p(B|A) = 60\%$

$A =$ 3J student

$B =$ male.

$\therefore p(A \cap B) = p(B|A) \cdot p(A)$

Total probability Rule.

$\boxed{\text{Exhaustive.}}$ in general, a collection of

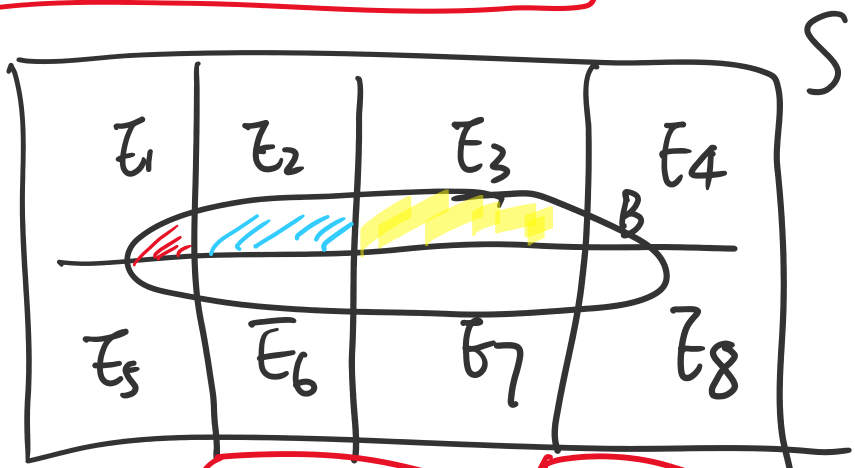
$E_1, E_2 \dots\dots E_k$

$E_1 \cup E_2 \cdots\cdots \cup E_k = S$

$E_1 \cap E_2 = \phi$, $E_2 \cap E_3 = \phi \cdots\cdots$ each pair are

$\boxed{\text{mutually exclusive.}}$

$k = 8.$



$\therefore p(B) = p(B \cap E_1) + p(B \cap E_2) + \cdots p(B \cap E_8)$

$\therefore p(B) = \underline{p(B\cap E_1)} + p(B\cap E_2) + \dots p(B\cap E_8)$

$= p(B|E_1)\cdot p(E_1) + p(B|E_2)p(E_2) \dots\dots p(B|E_8)\cdot p(E_8)$

Example:

A machine can fail due to internal reasons (93%) or External reasons (7%).

Internal reasons: A — C ① ~~Lack of M.Oil~~ 69%.

D ② Foreign objects. 31%.

External reasons B — E ① Falsely perfomly 53%

F ② Bad weather 27%

G ③ others 20%

$p(A) = 93\%$ $p(B) = 7\%$

$p(C|A) = 69\%$

$\therefore \boxed{\underline{p(C)} = p(C\cap A)} = p(C|A)\cdot p(A) = 93\%\cdot 69\%$

Example:

20% 3J 80% 3F.

60% male 40% Female 70% male 30% Female

what's the probability of overall Female.

$A =$ 3J students. $p(A) = 20\%$

A = 3J students. $p(A) = 20\%$

B = 3Y students. $p(B) = 80\%$

C = male

D = Female.

$p(C|A) = 60\%$ $p(C|B) = 70\%$

$p(D|A) = 40\%$ $p(D|B) = 30\%$

? $p(D)$

$A \cup B = S$

$A \cap B = \phi$

$p(D) = p(D|A) \cdot p(A) + p(D|B) \cdot p(B)$

$= 20\% \, 40\% + 80\% \cdot 30\%$

Example:

A_1, A_2, A_3, three different kinds of defaults we can have.

$p(A_1) = 0.39$ $p(A_1 \cup A_3) = 0.64$

$p(A_2) = 0.36$ $p(A_2 \cup A_3) = 0.69$

$p(A_3) = 0.39$ $p(A_1 \cap A_2 \cap A_3) = 0.04$.

$p(A_1 \cup A_2) = 0.61$

(a) exactly 2 of 3 types of defaults happen.

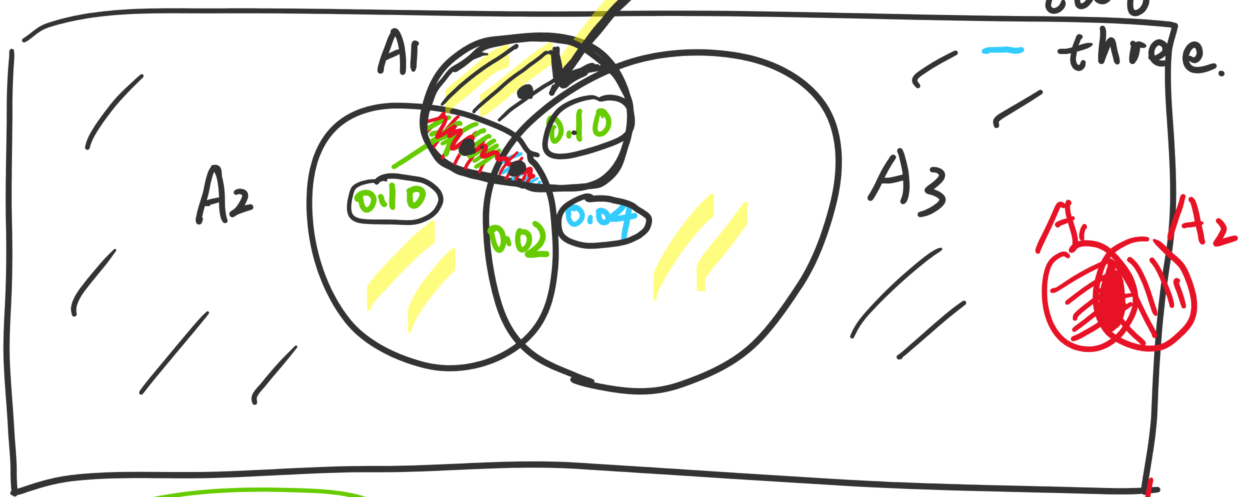
$p(A_1 \cap A_2) + p(A_1 \cap A_3) + p(A_2 \cap A_3)$

– nothing

$p(A_1 \cap A_2) + p(A_1 \cap A_3) + p(A_2 \cap A_3)$

$- 3p(A_1 \cap A_2 \cap A_3)$ $\boxed{A_1 \cap C'}$

— nothing
one
two
three.



$p(A_1 \cap A_2) = p(A_1) + p(A_2) - p(A_1 \cup A_2)$

$\Leftrightarrow p(A_1 \cup A_2) = p(A_1) + p(A_2) - p(A_1 \cap A_2)$

$= 0.14$

$p(A_1 \cap A_3) = 0.39 + 0.39 - 0.64 = 0.14.$

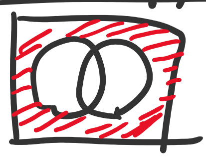
$p(A_2 \cap A_3) = 0.36 + 0.39 - 0.69 = 0.06.$

$0.14 - 0.04$

(a) $0.1 + 0.1 + 0.02 = 0.22$

(b) the probability that type 1 default (A_1) given that type 2 and type 3 are not happened.

$C = ((A_2 \cup A_3)')$ $\leftarrow$ A_2



$(A_2 \cup A_3)'$

$$p(A_1 | C) = \frac{p(A_1 \cap C)}{p(C)}$$

$$= \frac{0.39-0.1-0.1-0.04}{1-0.69}$$

$$= \frac{15}{31}.$$

2.6 Independence.

$\boxed{p(B|A) = p(B)} \Longleftrightarrow \underline{p(A \cap B) = p(A) \cdot p(B)}$

different

Mutualy Exclusive.

$p(A \cap B) = 0$

$\Rightarrow p(A \cap B) = p(B|A) \cdot p(A). = p(B) \cdot p(A).$

Example:

105 chips and 15 defective.

select 2. without replacement.

(a) The probability that the second is defective is ?

A ① D, D ←.

$\frac{15}{105}$ B ② N, D

$$p(A) = \frac{15}{105} \cdot \frac{14}{104}$$

$$p(A) = \frac{15}{105} \cdot \frac{14}{104}$$

$$p(B) = \frac{90}{105} \cdot \frac{15}{104}$$

$$p(A) + p(B) = \frac{15 \cdot 14 + 90 \cdot 15}{105 \cdot 104} = \frac{(\cancel{14+90})15}{105 \cdot \cancel{104}}$$

$$= \frac{15}{105}.$$

(b) If three are chosen, the first is defective and third is not.

$$\frac{15}{105} \cdot \frac{14}{104} \cdot \frac{90}{103} \quad D\ D\ N$$

$$+ \frac{15}{105} \cdot \frac{90}{104} \cdot \frac{89}{103} \quad D\ N\ N$$

$$= \frac{45}{364}.$$