Lecture 10

May 30, 2018 5:10 PM

$f(x) = 3x^2, \quad 0 \le x \le 1$

$n = 2$

$E(X) = \int x f(x)\,dx = \int 3x^3\,dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}$

$E(X^2) = \int x^2 f(x)\,dx = \int 3x^5\,dx = \frac{3}{5}x^5\Big|_0^1 = \frac{3}{5}$

$\therefore V(X) = E(X^2) - E(X)^2 = \frac{3}{5} - \frac{9}{16} = \frac{48-45}{80} = \frac{3}{80}$

$V(\bar{X}) = \frac{V(X)}{n} = \frac{3}{160}$

$V(\bar{X}) = V\left(\frac{X_1 + \cdots X_n}{n}\right) = \frac{1}{n^2}\underbrace{(V(X_1) + \cdots V(X_n))}_{n}$

$= \frac{1}{n^2} \cdot n \cdot \sigma^2$

$= \frac{\sigma^2}{n} = \frac{V(X)}{n}$

Chapter 9:

Hypothesis Test

C.I. Defination

1000 C.I., 95% CI

950 of C.I. contains the true mean.

A statistical hypothesis:

a statement about the parameters of the popopulation.

Hypothesis Test:

1. Hypothesis: $H_0: \mu = 100{,}000$ Average Income

$H_A: \mu < 100{,}000$

Null hypothesis: H_0 is the population parameter that equal to some value.

Alternative hypothesis: H_a (H_A) is that the that equal ...

Alternative hypothesis: H_a (H_A) is that the population parameter ">", "<", "≠" to some value.

e.g. $H_0: \mu = 170$, $H_A: \mu \neq 170$.

$H_0: \mu = 170$, $H_A: \mu > 170$.

2. Test Statistic:

$\bar{x}_1 = 50{,}000$, $\bar{x}_2 = 90{,}000$

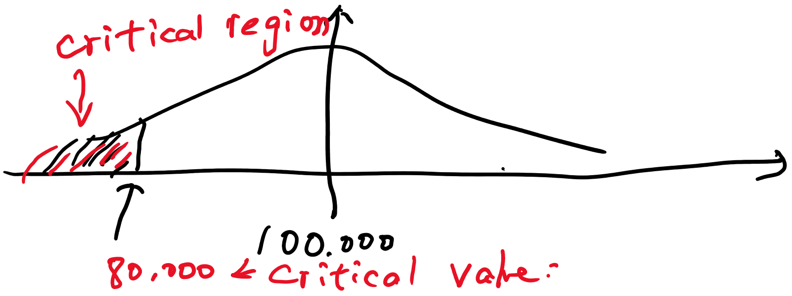
is a statistic: whose value is used to make a decision about the hypothesis.

3. Critical region, Critical value.

$H_0: \mu = 100{,}000$

$H_A: \mu < 100{,}000$

Test statistic: $\bar{x} = 50{,}000$



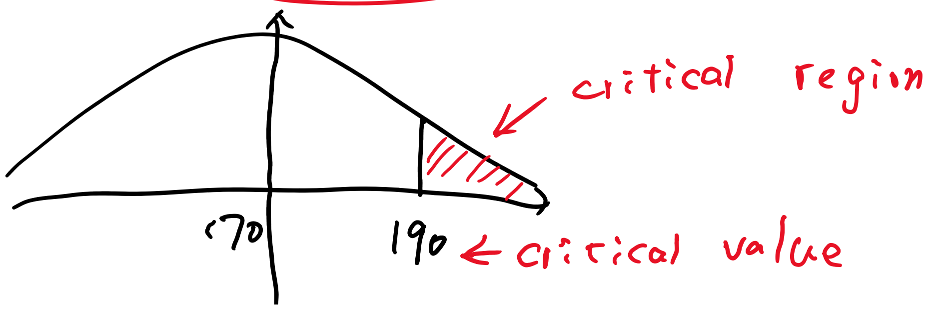
when

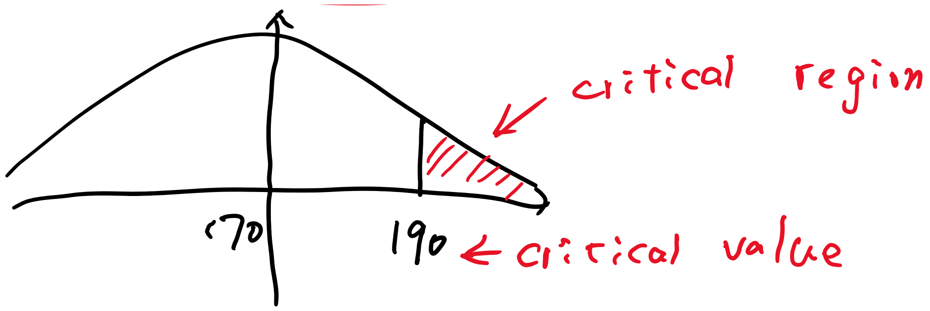
$\bar{x} = 50{,}000 < 80{,}000$

The critical region consists of values of the test statistic that resulting in rejecting the null hypothesis.

$H_0: \mu = 170$

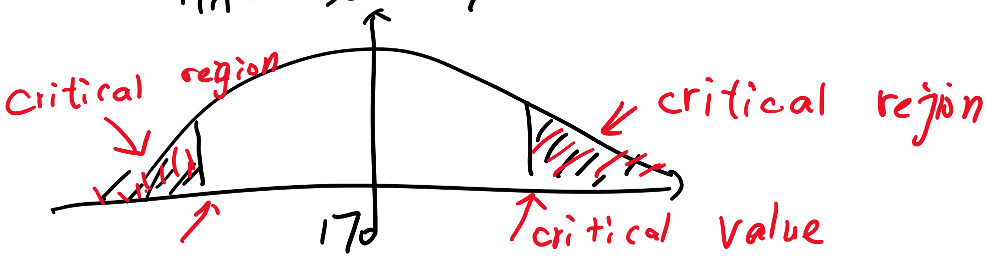
$H_A: \mu > 170$





$H_0: \mu = 170$

$H_A: \mu \neq 170$



4. Make a conclusion.

When Test statistic lies in the critical rejion, then reject the null hypothesis.
Otherwise, do not.

Type of errors:

	H_0 is True	H_A is True.
Reject H_0	Type I error	Correct
Accept H_0	Correct	Type II error

α: A type I error occurs if we reject H_0 but H_0 is actually true. (Patient with disease, fail to diagnosis)

β: A type II error occurs if we do not reject H_0 when H_A is actuall true (Healthy people diagnosed wrongly)

α : type I error, significance level or size of the test

β: type II error.

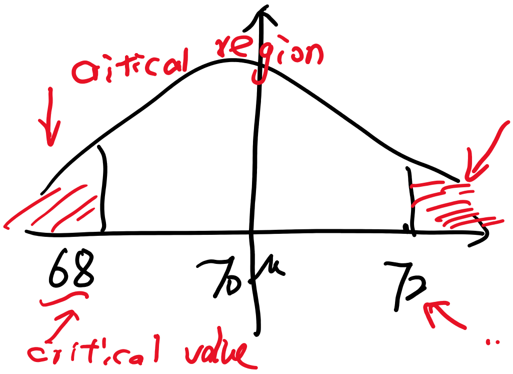
$1-\beta$: power of the test

Example: Sample $n = 64$ from a normal population.

$H_0: \mu = 70$

$H_A: \mu \neq 70$

Suppose we reject H_0 if $\bar{x} \geq 72$ or $\bar{x} \leq 68$. Find the α, assuming $\sigma = 16$.



$$\alpha = \text{Type I error}$$

$$= P(\text{rejecting } H_0 \mid \underbrace{H_0 \text{ is actually true}}_{\mu = 70})$$

$$= P(\bar{x} \leq 68 \text{ or } \bar{x} \geq 72)$$

$$= P(\bar{x} \leq 68) + P(\bar{x} \geq 72)$$

$\mu = 70$

$$= P\left(Z \leq \frac{68-70}{\underset{\sigma}{16}/\sqrt{\underset{n}{64}}}\right) + P\left(Z \geq \frac{72-70}{16/\sqrt{64}}\right)$$

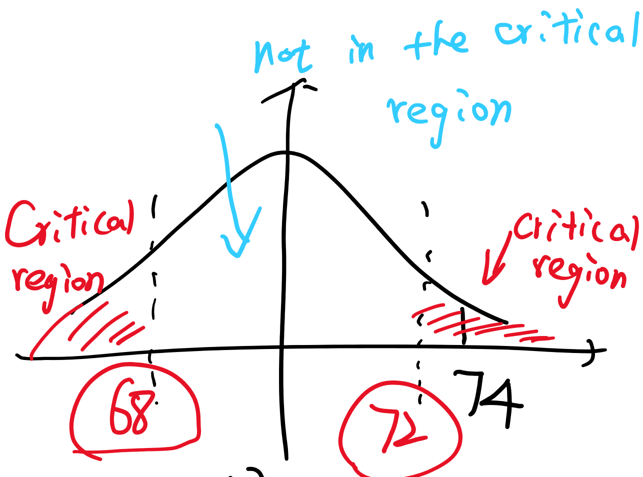
$$= P(Z \leq -1) + P(Z \geq 1)$$

$$= 2P(Z \leq -1)$$

$$= 2 \cdot 0.1587$$

β, additional condition

true mean: $\mu = 74$



$$\beta = \text{type II error}$$

$$= P(\text{fail to reject } H_0 \mid H_A \text{ is correct})$$

$= P(\text{fail to reject } H_0 \mid H_A \text{ is correct})$

$= P(68 \le \bar{x} \le 72 \mid \mu = 74)$

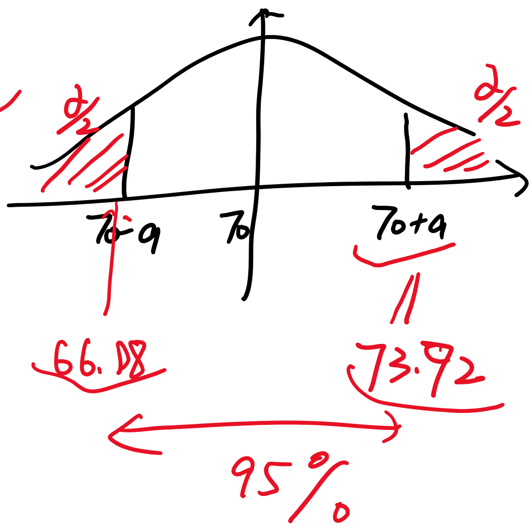
$= P\left(\frac{68-74}{16/\sqrt{64}} \le Z \le \frac{72-74}{16/\sqrt{64}}\right)$

$= P(-3 \le Z \le -1)$

$= 0.9986 - 0.8413$

$= 0.1573$

Example: Find critical value, $\alpha = 5\%$



$\therefore \underbrace{P(\bar{x} \ge 70+a) + P(\bar{x} \le 70-a)}$

$= P\left(Z \ge \frac{70+a-70}{16/\sqrt{64}}\right) + P\left(Z \le \frac{70-a-70}{16/\sqrt{64}}\right)$

$= P(Z \ge \frac{a}{2}) + P(Z \le \frac{-a}{2})$

$= 5\% = \alpha$

$= 2 \cdot P(Z \le \frac{-a}{2})$

$\therefore P(Z \le \frac{-a}{2}) = 0.025$

$\therefore \frac{a}{2} = 1.96 \Rightarrow a = 3.92$

In general: (Relationship between C.I and Hypothesis Test)

When testing $H_0: \mu = \mu_0$ / $H_A: \mu \ne \mu_0$

we reject H_0 at α (significance level)

$\Longleftrightarrow$ the $100(1-\alpha)\%$ CI for average does not contain the true mean μ.

Comment: 1. n is fixed, $\alpha \uparrow \Rightarrow \beta \downarrow$

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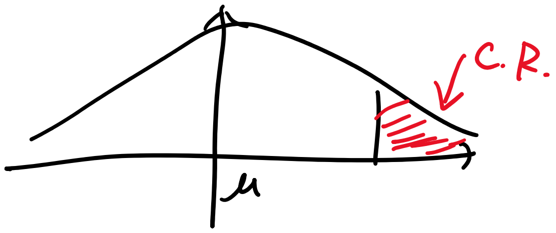
$\beta \uparrow \Rightarrow \alpha \downarrow$

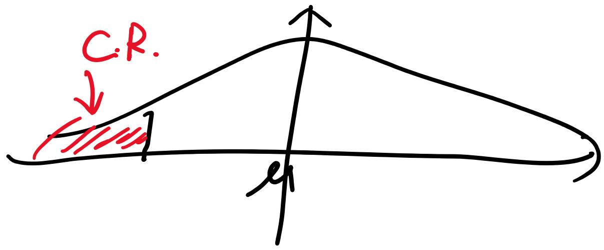
2. Increasing n, reduce α, β.

One-sided Hypothesis Test:

1. $\begin{cases} H_0: \mu = 70 \\ H_A: \mu > 70 \end{cases} \Longrightarrow$



2. $\begin{cases} H_0: \mu = 70 \\ H_A: \mu < 70 \end{cases}$



Example: Suppose that

Test: A coin is biased and having larger chance for head.

Flip it 10 times and if find 8 heads or more

then we claim is biased.

$\begin{cases} H_0: p = \frac{1}{2} \\ H_A: p > \frac{1}{2} \end{cases}$

$\text{Binomial}(n=10, \underbrace{p=\tfrac{1}{2}}_{H_0 \text{ is true}})$

(1) Find α.

$\alpha = P(\text{reject } H_0 \mid \underbrace{H_0 \text{ true}}_{p=\frac{1}{2}})$

$= P(X \geq 8 \mid p = \frac{1}{2})$

$= \binom{10}{8}(\frac{1}{2})^8(\frac{1}{2})^2 + \binom{10}{9}(\frac{1}{2})^9(\frac{1}{2}) + \binom{10}{10}(\frac{1}{2})^{10}(\frac{1}{2})^0$

$= 0.0547$

(2) Find β if $p = \frac{2}{3}$

$\beta = P(\text{not reject } H_0 \mid \underline{H_A \text{ is true}})$

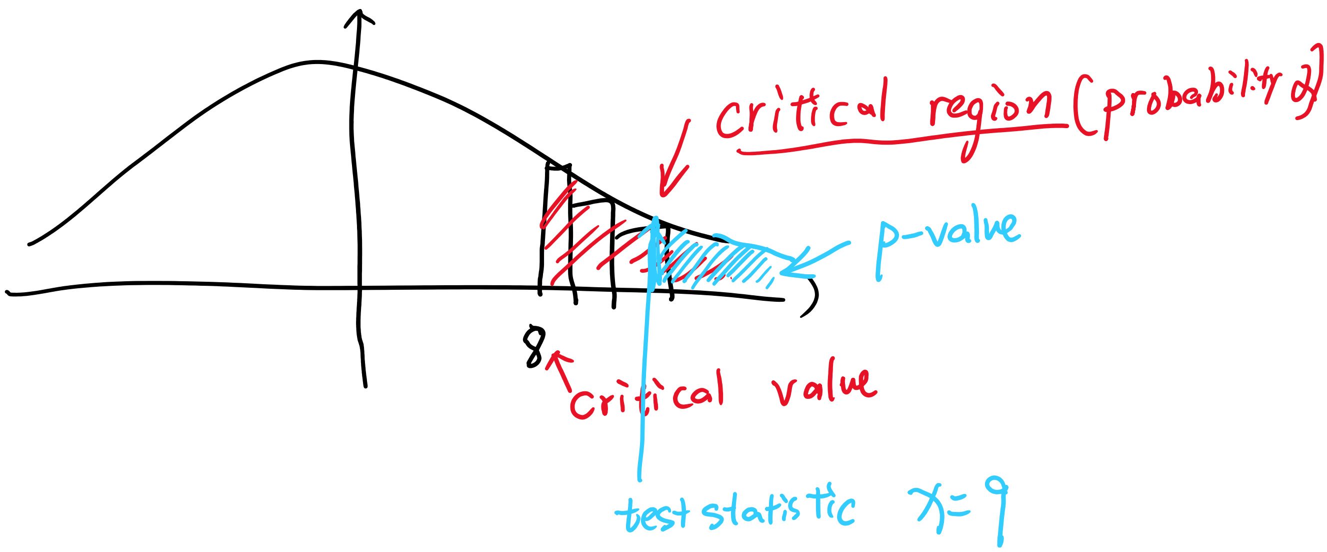
$$\beta = P(\text{not reject } H_0 \mid \underbrace{H_A \text{ is true}}_{p=\frac{2}{3}})$$

$$= P\left(X \le 7 \mid p = \tfrac{2}{3}\right)$$

$$= 1 - \left[\binom{10}{8}\left(\tfrac{2}{3}\right)^8\left(\tfrac{1}{3}\right)^2 + \binom{10}{9}\left(\tfrac{2}{3}\right)^9\left(\tfrac{1}{3}\right) + \binom{10}{10}\left(\tfrac{2}{3}\right)^{10}\left(\tfrac{1}{3}\right)^0\right]$$

$$= 0.7009$$

p-value is the probability that obtain a value of test statistic at least as extreme as the observed value when H_0 is true.



$\therefore$ p-value $< \alpha \Rightarrow$ reject H_0
p-value $> \alpha \Rightarrow$ do not reject H_0

(c) Suppose we have an observation $X = 6$.

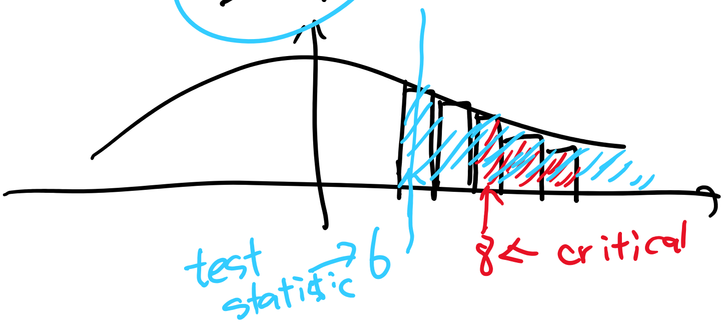
Find the p-value.

$$\therefore \text{p-value} = P(\underbrace{X = 6 \text{ or more}}_{\text{tail probability}} \mid \underbrace{p = \tfrac{1}{2}}_{H_0 \text{ is true}})$$

$$\binom{10}{6}\left(\tfrac{1}{2}\right)^6\left(\tfrac{1}{2}\right)^4 + \cdots + \binom{10}{10}\left(\tfrac{1}{2}\right)^{10}\left(\tfrac{1}{2}\right)^0$$

$$= \binom{10}{6}\left(\tfrac{1}{2}\right)^6 \cdot \left(\tfrac{1}{2}\right)^4 + \cdots \binom{10}{10}\left(\tfrac{1}{2}\right)^{10}\left(\tfrac{1}{2}\right)^0$$

$> \alpha \Rightarrow$ do not reject H_0



$P(X \geq 6) = $ p-value

$P(X \geq 8) = \alpha$

1. Tests on the mean μ of normal distribution. | σ is known.

Sample: $X_1 \ldots X_n$ with mean μ and variance σ^2.

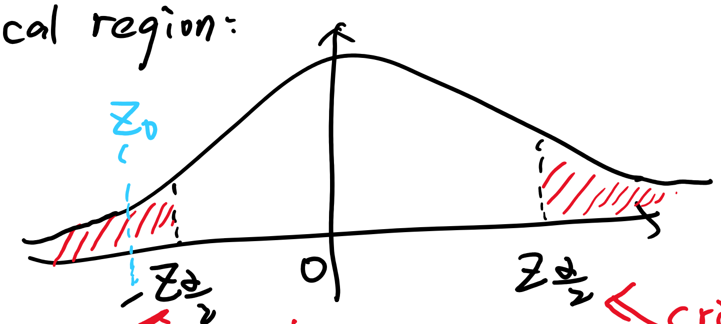
Then, we want to test

$$\begin{cases} H_0: \mu = \mu_0 \\ H_A: \mu \neq \mu_0 \end{cases}$$

Test statistic:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

Critical region:



Then the p-value:

$= \boxed{2P(Z < Z_0)}$ then compare with α.

... of reject H_0 | H_0 is true)

$\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true})$
$\mu = \mu_0$

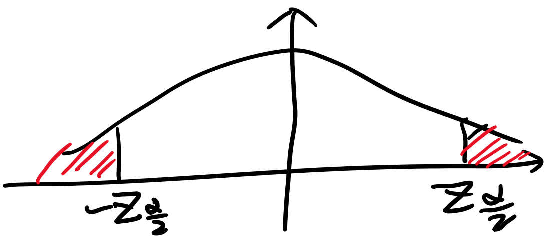
Conclusion:

Reject H_0, then test is significant, we could conclude that H_A is correct.

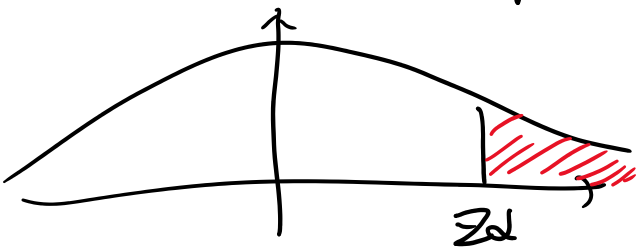
Not reject H_0, the test is not conclusive, we can not prove H_0 is wrong based on the sample.

When to reject ?

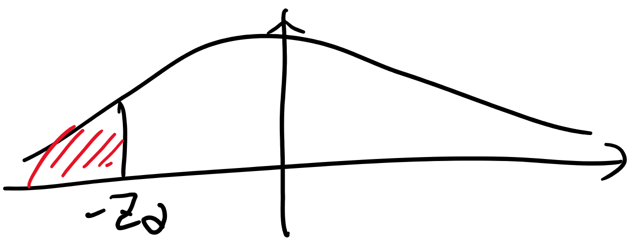
1. $H_1: \mu \neq \mu_0$, reject H_0 at α if $Z > Z_{\frac{\alpha}{2}}$ or $Z < -Z_{\frac{\alpha}{2}}$



2. $H_1: \mu > \mu_0$, reject H_0 at α if $Z > Z_\alpha$



3. $H_1: \mu < \mu_0$, reject H_0 at α if $Z < -Z_\alpha$



or when p-value is less α, reject H_0
p-value $< \alpha$

Example: $\mu = 98.6$ to test

$n = 106$, $\bar{x} = 98.2$ then to test whether the true mean

$n=106$, $\bar{x}=98.2$ then to test whether the true mean is 98.6 or not. σ is given as 0.62. ($\alpha=0.01$)

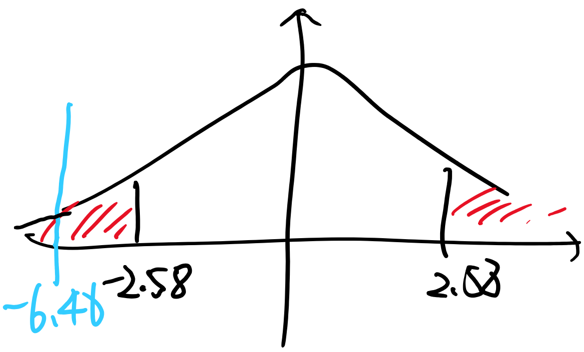
1. Hypothesis:
$$\begin{cases} H_0: \mu = 98.6 \\ H_A: \mu \neq 98.6 \end{cases}$$

2. Test statistic:
$$Z = \frac{98.2 - 98.6}{0.62/\sqrt{106}} = -6.64.$$

3. Critical value:

① two tail, $\alpha = 0.01$

$$Z_{\frac{\alpha}{2}} = Z_{0.005} = 2.58$$
$$-Z_{\frac{\alpha}{2}} = -Z_{0.005} = -2.58$$



② $p\text{-value} = P(Z < -6.64) \cdot 2 < 0.01$

4. Conclusion:

As $Z < -Z_{\frac{\alpha}{2}}$, then we reject the H_0 and claim that the true mean may not be 98.6.

Type II error: β: $\boxed{\mu = \mu_0 + \delta}$ difference

true mean

mean in H_0

($\delta \rightarrow$ delta
$\sigma \rightarrow$ sigma)

$$\underbrace{\beta = P\left(Z \le Z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}\right) - P\left(Z \le -Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)}_{\text{From CLT.}}$$

Choice of sample size:

$$\frac{(Z_{\alpha} + Z_{\beta})^2}{} \quad \sigma^2 \quad , \quad \delta = \mu - \mu_0$$

$$n = \frac{(Z_{\frac{\alpha}{2}} + Z_{\beta})^2}{\sigma^2} \delta^2 \quad \text{wher} \quad \delta = \mu - \mu_0$$

For given α, β.

2. Test on the mean of Normal when Variance Unknown

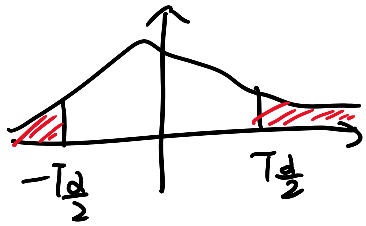
μ $\qquad\qquad$ σ unknown

1. Hypothesis: $H_0: \mu = \mu_0 \qquad H_A: \mu \neq \mu_0$
2. Test Statistic:

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$$

3. Critical region

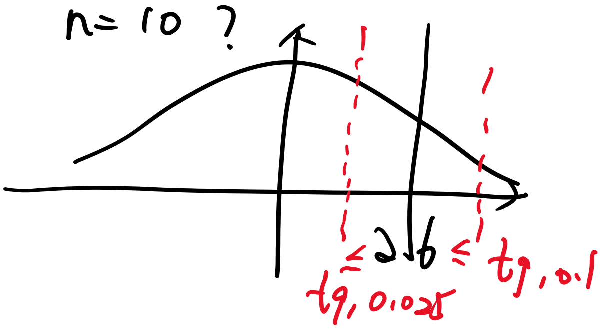
$$-T_{\frac{\alpha}{2}}, \quad T_{\frac{\alpha}{2}}$$



4. Conclusion.
   1. reject if p-value $< \alpha$
   2. T (test statistic) in the critical region.

T table can not help you find p-value exactly.

Example: $T = 2.6$ (test statistic) one tail ($\mu > \mu_0$)
what's p-value if $n = 10$?



$t_{9,0.1} = 2.821$

$t_{9,0.025} = 2.237$

$0.025 < \text{p-value} < 0.1$

0.025 = p ...

3. Tests on population proportions:

1 Hypothesis: $H_0: p = p_0$

$H_A: p \neq p_0$

2. Test statistic

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \quad \text{where } \hat{p} = \frac{x}{n}$$

3. critical region. (Similar to μ with σ known)

4. Conclusion.

Type II Error: ① when $p < p_0$

$$\beta = 1 - P\left(Z \le \frac{p_0 - p - z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p_0(1-p_0)n}}\right)$$

a

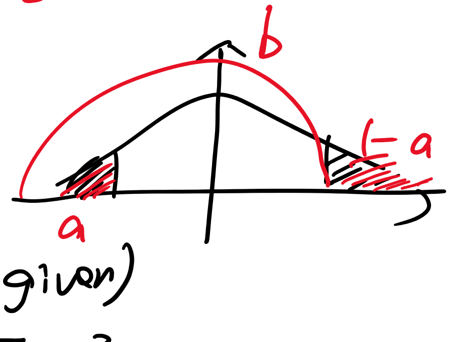
② when $H_1: p > p_0$

$$\beta = P\left(Z \le \frac{p_0 - p + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p_0(1-p_0)/n}}\right)$$

b

③ $H_1: p \neq p_0$

$$\beta = b - a$$



Choice of sample size. (α, β given)

$$n = \left[\frac{z_\alpha\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p_0(1-p_0)}}{p - p_0}\right]^2 \quad \text{(one tail)}$$

Chapter 10: Difference between means from Normal population

Chapter 10: Difference between ... population

4. Tests on μ_1 and μ_2 when assuming equal variance ($\sigma_1^2 = \sigma_2^2$) assuming σ_1, σ_2 is known

1. Hypothesis: $H_0: \mu_1 = \mu_2$

$$H_A: \begin{cases} \mu_1 \neq \mu_2 \\ \mu_1 > \mu_2 \\ \mu_1 < \mu_2 \end{cases}$$

2. Test statistic:

$$Z = \frac{\bar{X}_1 - \bar{X}_2 - \overset{0}{(\mu_1 - \mu_2)}}{\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0,1)$$

where $S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$ $\left(\underset{\uparrow\ \sigma_1}{\overset{SD(\bar{x}_1)}{S_1}},\ \underset{\uparrow\ \sigma_2}{\overset{SD(\bar{x}_2)}{S_2}} \right)$

pooled sample variance.

3. Critical Reigion

4. Conclusion.

5. Tests on μ_1 and μ_2 assuming σ unknown but equal variance ($\sigma_1^2 = \sigma_2^2$)

Test statistic:

$$T = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t(\nu)$$

pivot

$$\nu = n_1 + n_2 - 2$$

C.I for $(\mu_1 - \mu_2)$

$$\bar{x}_1 - \bar{x}_2 \pm t_{n_1+n_2-2,\,\alpha/2}\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

6. Tests on μ_1 and μ_2 assuming variance unknown but unequal variance.

Test statistic:

$$T = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$\sim t(\nu)$$

Degree of freedom:

$$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

C.I:

$$\bar{x}_1 - \bar{x}_2 \pm t_{\nu,\,\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Example:

Two supplies manufacture a plastic gear used in a laser printer.

Supplier 1 had a mean impact $\bar{x}_1 = 290$

$s_1 = 12$, $n_1 = 10$.

Supplier 2 : $\bar{x}_2 = 321$, $s_2 = 22$, $n_2 = 16$.

Assume that population variances are unknown and not equal. $\alpha = 0.05$.

Assume that pop...

not equal. $\alpha = 0.05$.

Test the hypothesis that two suppliers have difference in mean impact.

two tail

1. Hypothesis:

$$H_0: \mu_1 = \mu_2$$

$$H_A: \mu_1 \neq \mu_2$$

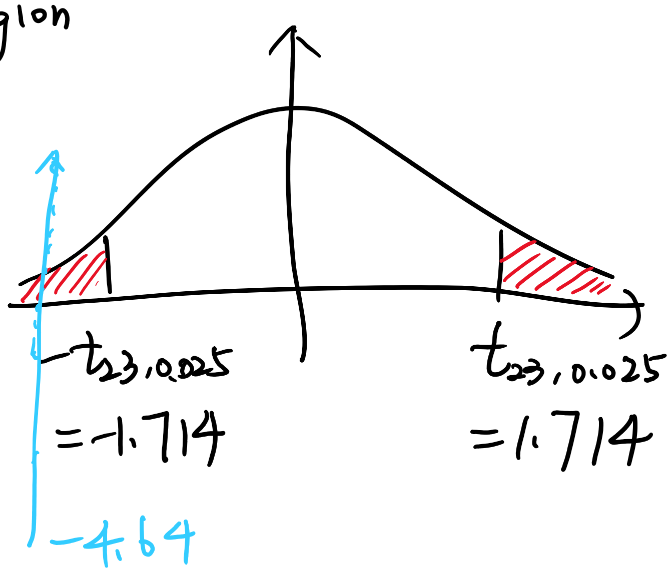
2. Test statistic:

$$T = \frac{290 - 321}{\sqrt{\frac{12^2}{10} + \frac{22^2}{16}}} = -4.64 \sim t(\nu)$$

$$\sim t(23)$$

$$\nu = \frac{\left(\frac{12^2}{10} + \frac{22^2}{16}\right)^2}{\frac{(12^2/10)^2}{10-1} + \frac{(22^2/16)^2}{16-1}} \approx 23.72 \approx 23$$

3. Critical Region



$$\therefore T = -4.64 < -t_{23,0.025} = -1.714$$

$\Rightarrow$ reject H_0.

4. Conclusion: the test is significant and we could conclude that they have different mean of impact.

C.I. 95%

$$t_{23,0.025} \text{ (two tail } \alpha = 5\%) \quad \sqrt{\frac{12^2}{\ } \quad \frac{22^2}{\ }}$$

t25, 0.025

C.I. 95%

$$(290 - 321) \pm 1.714\sqrt{\frac{12^2}{10} + \frac{22^2}{16}}$$

$$= (-42.45, -19.55)$$