Tutorial 2 Suggested Solutions

Probability

Sept. 23

Page 34 - Example 5d,e,i

Example 5d

An urn contains n balls, one of which is special. If k of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

Solution: To find the probability that the special ball is selected we need:

$$P(\text{special ball is selected}) = \frac{\text{number of ways special ball is selected}}{\text{number of ways in total to select the } k \text{ balls}}$$

To find the number of ways special ball is selected we need to subset the n balls. There is 1 special ball and thus there are $n-1$ not special balls. We want to make sure to select the special ball, this can be done $\binom{1}{1}$. And we still need to select a total of k balls, thus of the remaining $n-1$ not special balls we need to select $k-1$ of them. Thus there are $\binom{n-1}{k-1}$ ways to select the remaining balls.

For the denominator we want to total number of ways to select k balls from a possible n. This can be done in $\binom{n}{k}$ ways.

Subbing in the solution for the numerator and denominator:

$$P(\text{special ball is selected}) = \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

Alternatively, we can think that we are selecting the balls successively and have looked at the event that the special ball is the i^{th} ball selected. For $i = 1, ..., k$. Then, since each one of the n balls is equally likely to be the i^{th} ball selected, it follows that $P(\text{the special ball is the } i^{th} \text{ ball selected})$ = 1/n. Then we add up all the cases for $i = 1, ..., k$.

$$P(\text{special ball is selected}) = \Sigma_{i=1}^{k} P(\text{the special ball is the } i^{th} \text{ ball selected})$$

$$P(\text{special ball is selected}) = \Sigma_{i=1}^{k} \tfrac{1}{n} = \tfrac{k}{n}$$

Example 5e

Suppose that $n+m$ balls, of which n are red and m are blue, are arranged in a linear order in such a way that all $(n+m)!$ possible orderings are equally likely. If we record the result of this experiment by listing only the colors of the successive balls, show that all the possible results remain equally likely.

Solution: Consider any one of the $(n+m)!$ possible orderings, and note that any permutation of the red balls among themselves and of the blue balls among themselves does not change the sequence of colors. For example is we had $\{R_1, B_1, B_2, R_2, R_3\}$ and decided to swap or rearrange the red balls, this does not affect the sequence of colours. Likewise if we decided to swap or rearrange the blue balls, this also does not affect the sequence of colours. As a result, every ordering of colorings corresponds to $n!\ m!$ different orderings of the $n+m$ balls, so every ordering of the colors has probability $\frac{n!m!}{(n+m)!}$ of occurring.

Thinking about the example: $\{R_1, B_1, B_2, R_2, R_3\}$. There are $m = 2$ blue balls and $n = 3$ red balls. There are 2! ways to swap/rearrange the blues, and there are 3! ways to rearrange the red balls. In total there are 5! = (2+3)! ways of rearranging all of the balls. Thus the probability of having the sequence $\{R_1, B_1, B_2, R_2, R_3\}$ is $\frac{2!3!}{5!}$

Example 5i (Slightly Modified)

If n people are present in a room, what is the probability that at least two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is above than 0.5?

Solution:

We want to probability at least 2 people share the same birthday. If we try this calculation directly it is a bit cumbersome. Thus we can use the compliment.

$$P(\text{at least 2 people share the same birthday}) = 1 - P(\text{no 2 people share the same birthday})$$

$$= 1 - P(\text{everybody has different birthdays})$$

$$= 1 - \frac{\text{number of ways everybody has different birthdays}}{\text{total number of ways for } n \text{ birthdays}}$$

For the denominator: Since each person can celebrate his/her birthday on any one of 365 days there is a total of $(365)^n$ possible outcomes.

For the numerator: To ensure that everyone has a different birthday the first person can be born on any of the 365 days. The second person can be born on any of the 364 days left (after ruling out the first persons birthday). The third person can be born on any of the 363 days left (after ruling out the first and second persons' birthdays). ... The n^{th} person can be born on any of the $365 - (n-1)$ days left (after ruling out the first $n-1$ persons' birthdays). Thus the number of ways to ensure that these n people all have different birthdays is: (365)(364)(363)...(365-$(n-1)$).

$$P(\text{at least 2 people share the same birthday}) = 1 - \frac{(365)(364)(363)...(365-(n-1)}{365^n}$$

By plugging in different values of n, one can see that for $n \geq 23$ the probability of at least 2 people share the same birthday is more than 0.5.

Page 52 - Theoretical Exercise 4 - part 1

Exercise 4 - Part 1

$\left(\bigcup_{i=1}^{\infty} E_i\right) F = \bigcup_{i=1}^{\infty} (E_i F)$

Solution:
From distributive law for the sets E_1, E_2, and F, we have:

$$(E_1 \cup E_2)F = E_1F \cup E_2F$$

We can now extend this. So instead of E_1, E_2, we have infinitely many sets $E_1, E_2, E_3, ...$
Thus:

$$\begin{aligned}\Big(\bigcup_{i=1}^{\infty} E_i\Big)F &= (E_1 \cup E_2 \cup E_3 ...)F \\ &= E_1F \cup E_2F \cup E_3F \cup ... \\ &= \bigcup_{i=1}^{\infty}(E_iF) \quad \blacksquare\end{aligned}$$

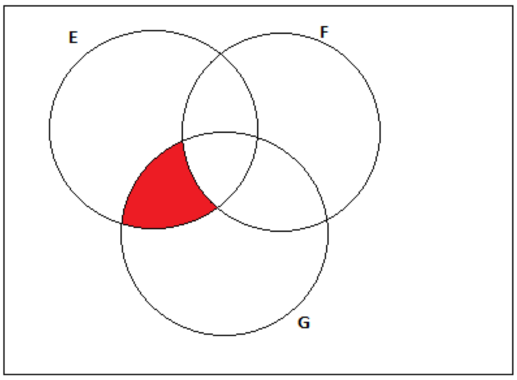
Page 52 - Theoretical Exercise 6b,d,h

Let E, F, and G be three events. Find expressions for the events so that, of E, F, and G:

Exercise 6b

Both E and G, but not F occur.

Solution: We want the shaded region in the Venn Diagram.



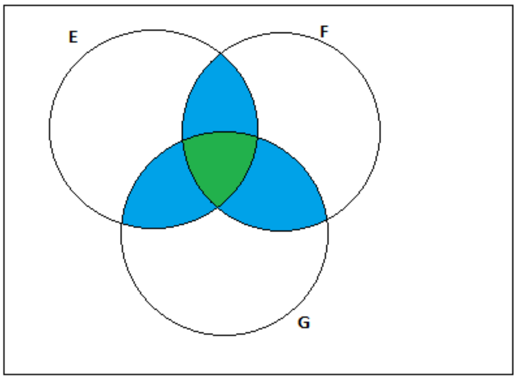
We can see that the red region is in the intersection of both E and G but does not include F. Thus we can denote this as:

$$EGF^c$$

Exercise 6d

at least two of the events occur.

Solution: Since we want at least two of the events to occur, this means we want either two of the events to occur simultaneously (blue), or we three events to occur simultaneously (green).



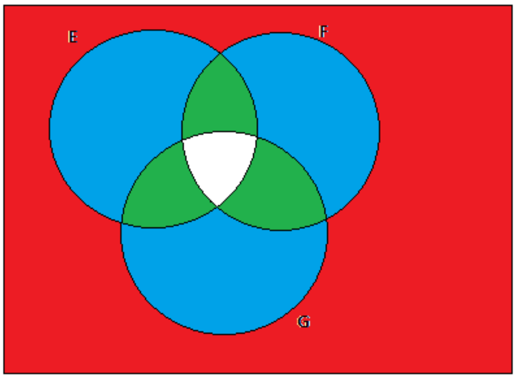
We can see that each blue region combined with the green region makes up an interection of two events. If we take the union of the intersections of these pairwise intersections we will be left with the entire region. Thus we can denote this as:

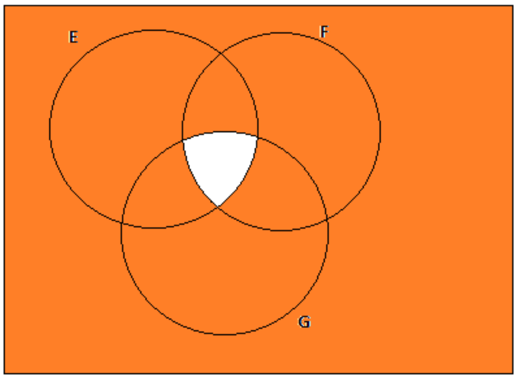
$$EF \cup EG \cup FG$$

Exercise 6h

at most two of the events occur.

Solution: Since we want at most two of the events to occur, this means we want either 0(red) or 1(blue) or 2(green) of the events to occur simultaneously. Thus we want to denote the entire shaded region.





We notice that this region contains everything in the entire sample space except for the triple intersection. Thus, the shaded region is the complement of the intersection of E, F, and G. Thus we can denote this as:

$$(EFG)^c$$

Solutions for 6 may vary.