Tutorial 1 Suggested Solutions

Counting, Permutations, & Combinations

Sept. 16

1 Tutorial Info

Check back every Friday for the week's solutions. Enjoy your weekend, and see you next week.

2 Lozinski's Puzzle

2.1 Random Guessing

Since there are 6 pieces, and each piece can have 2 orientations (flipped vertically). That piece of information gives us 2^6 choices.

Since the order definitely does matter in this case (why?), there are 6 pieces which we can pick for the first slot (or whichever slot you want). For the second slot (or whichever other *unused* slot) there are 5 pieces which can be selected. This gives us $6 \cdot 5 \cdots 2 \cdot 1 = 6!$

Thus by the "Basic Principle of Counting" we have

 $2^6 \cdot 6! = 46080$

2.2 Using Intuition

Edges: An "=" must lie somewhere in between two expressions, which means the two strips without the = must be at the ends. As before, each of them have 2 orientations, and there are 2 of them. Hence we get: $2^2 \cdot 2! = 8$ ways to arrange our ends.

Middle: The remaining 4 pieces must go in the middle: for now, this gives us 4! arrangements. We must not have two = signs in the same line however, since some rows will then have no = signs. That means we will have $2! \cdot 2!$ ways to arrange them: the first and fourth rows must each take one of the =, so the first row can take 1 or 2 pieces, the fourth row must then take the other remaining 1 (same logic for the second and third rows).

Hence, altogether we have

$$(2^2 \cdot 2!) \cdot (4! \cdot 2! \cdot 2!) = 4! \cdot 2^5 = 768$$

Notice: The Edges, and Middle are independent of each other in this case. How does that help us organize our thinking? (See how the brackets above group our expression)

3 Text Questions (Ross)

3.1 Binomial Theorem

3.1.1 Example 4d (page 8)

Let's look at the question in another way to avoid simply regurgitating **Theorem 4.2**:

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

We see we must multiply the binomial (x + y) by itself 3 times. Which means each time we do, there will be some "triplet" of x and y. The possibilities are

 $\{x, x, x\}, \{x, x, y\}, \{x, y, x\}, \{y, x, x\}, \dots$

and by symmetry, reverse the x's and y's.

There is only 1 way to get only x's or only y's. There are 3 ways to arrange x, x, and y.

$$(x+y)(x+y)(x+y) = 1 \cdot x^3 y^0 + 3 \cdot x^2 y^1 + 3 \cdot x^1 y^2 + 1 \cdot x^0 y^3$$

Hence we come to our *binomial coefficient*. There are many ways to express it, but we'll stick to the contemporary way.

$$\binom{n}{k} = \binom{n}{n-k} = \binom{n}{k, (n-k)} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdots (n-(k-1))}{k!}$$

Do you understand why there is (n - k + 1) in the numerator?

From *n* objects, we are choosing *k*. Or not choosing n - k. The important thing, is that we are splitting *x* and *y* into two groups. It does not matter in what order in this case. So in our case where n = 3 and k = 2 (for choosing 2 *x*'s, or choosing 1 *y*), we get $\binom{3}{2} = \binom{3}{1} = 3$.

Chapter 1.5 in the textbook discusses Multinomial Coefficients. $\binom{n}{k,(n-k)}$ is then another way of expressing the binomial coefficient in the more general multinomial coefficient form.

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$
, $n_1 + n_2 + \dots + n_r = n_r$

In our case, we have r = 2, and $n_1 = k$, $n_2 = (n - k)$. Where it is simple to see that k + (n - k) = n.

Hence, if it wasn't clear before, that is what the book means by

$$(x+y)^{3} = \binom{3}{0}y^{3} + \binom{3}{1}xy^{2} + \binom{3}{2}x^{2}y + \binom{3}{3}x^{3}$$

and where the general formula comes from (make sure you attempt the algebraic proof on your own):

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

There are other ways to visualize this solution. The beauty of mathematics is how ideas echo across multiple fields. There are different ways of coming up with this, but I'll bring up one in particular. If you've never seen *Pascal's Triangle*, I invite you to take a look at it below, and see if you recognize any patterns.



Figure 1: Pascal's Triangle – How does this relate to the binomial theorem?

A friendly warning, there will be **a lot** of things to memorize in statistics and probability if you don't take the time to understand **how** things are constructed. For Test 2, you will be required to be familiar with the various probability distributions (see inside the front and back covers) - either through brute-force rote memorization, or by understanding it intuitively.

3.1.2 Example 4e (page 8)

Did you notice that there were $8 = 2^3$ possibilities for our triplets in the last example? Firstly, let's understand the jargon of the question. Set theory plays a big role in mathematics in general, and the branch of probability is not exempt from this.

A set of n elements just means you have some arbitrary set, containing n objects, such as this arbitrary set X:

$$X = \{X_1, X_2, X_3, \dots, X_n\}$$

This is an abstract space.

A *subset* is a set where all elements are contained in another set. Very simply, let's look at some examples to grasp the basics of this definition:

$$\{X1, X2\} \subseteq X$$
$$\emptyset \subseteq X$$
$$X \subseteq X$$

It is important to observe that the empty set, denoted by \emptyset , and set itself are both subsets of X. (X is not a *proper subset* of itself).

Imagine $\{X_1, X_2, X_3, \ldots, X_n\}$ as a string of switches that can be flipped on or off (like in a computer, a string of 0s or 1s). It becomes analogous to our earlier problem that there are then

$$2 \cdot 2 \cdots 2 = 2^n$$

possibilities for subsets.

3.2 Self-Test Problems and Exercises

3.2.1 self-test problem 1.3

Read the question carefully, one innocently-seeming word can make all the difference. There are 3 DISTINCT positions.

a) Any of the $10 \rightarrow$ Presidente, Any of the remaining $9 \rightarrow$ Treasurer, Any of the remaining $8 \rightarrow$ Secretary. Thus we get

 $10\cdot 9\cdot 8$

for our final answer. Why is $\binom{10}{3}$ incorrect here? Hint: See above underlined.

b) Since the book does a fine job explaining the solution, I'll show you an alternate form. Everything is derived from the counting principle, so whichever method you use, as long as your philosophy is correct, you will attain the same answer.

Let's first note the stipulation. A and B will NOT work together, the question did not state that one of them had to work. So firstly, let's just *choose* a group of 3 from the remaining 8 mature individuals. But remember, the jobs are distinct, so there are 3! ways to arrange the 3 people who are chosen.

$$3!\binom{8}{3}$$

What if we do choose one of the 2 people though? He would then have 3 possible positions: $\binom{2}{1} \cdot 3$.

Let's then select 2 people from the remaining 8, and take into account that they can take either of the 2 roles: $2!\binom{8}{2}$. Thus multiplied together: $\binom{2}{1} \cdot 3 \cdot \binom{2!\binom{8}{2}}{=} = 3!\binom{2}{1}\binom{8}{2}$. Since these sets are independent, we will sum them together to get the total:

$$3!\binom{8}{3} + 3!\binom{2}{1}\binom{8}{2}$$

and with a little algebra, we get the same solution as the textbook: 672.

c) Use the same logic. Except this time, C and D must be together always (in or out of the committee). The stipulation on A and B does NOT carry over to this question.

C,D are out: See previous answers: $3!\binom{8}{3} = 8 \cdot 7 \cdot 6$

C,D are in: Here I will explain the book's $3 \cdot 2 \cdot 8$ logic. Since they must be together, let's treat them as ONE entity (that occupies two spots). Notice, $\binom{2}{2} = 1$ as there is only one way to choose everything from a set.

C can have any of the 3 spots, then D can have any of the 2 remaining spots. (*Why do we not have to multiply this by 2?*). For the final spot, there are 8 people waiting. Thus, we get the required: $3 \cdot 2 \cdot 8$.

Finally, sum up the parts, as seen in the text.

d) E must take one of 3 positions. Then as before, assign the remaining 2 distinct positions to the other 9 people $\rightarrow 9 \cdot 8$. Which explains the answer in the text.

e) F MUST be president. Effectively, makes the problem much simpler, since now we have a group of 9 people vying for 2 distinct positions: $2!\binom{9}{2} = 9 \cdot 8$.

However, if F is off the council, we use the same idea as before to arrange 3 distinct positions

for 9 people. The final answer, taking into account both possibilities, is then:

$$9 \cdot 8 + 3! \binom{9}{3} = 576$$

Take home point: Whichever method you choose, make sure you understand why you are using it. They both come from the same principle, but work differently. They will both lead to the same unique answer if you are correct, so if you have time on the test, use this to check your work.

3.2.2 self-test problem 1.4

The first part of the question should be obvious by now, as the wording gives away the answer: $\binom{10}{7}$.

For the more interesting question: She must answer at least 3 from the first 5, meaning we have 3 possibilities. She can answer (3, 4), (4, 3), or (5, 2). Remember, if we choose 3 problems from 5, we are consequently NOT choosing 2. This is why the binomial coefficient is being used, and we get our final answer of:

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 2(5\binom{5}{3}) + \binom{5}{2} = 110$$

3.2.3 self-test problem 1.6

For a license plate of the form: 1 A 2 B 3 C 4, we have 3 letters, and 4 digits (decimal system) in no particular formation (don't get confused with the Ontario plates). Thus, we get $26^3 \cdot 10^4$ possibilities.

We must also worry about the positions of the letters/digits. There are 7! permutations, however, we need to account for the digits and letters.

$$\frac{7!}{3!4!} = \frac{7!}{3!(7-3)!}$$

There is no coincidence that this matches the binomial coefficient:

$$\begin{pmatrix} 7\\ 3 \end{pmatrix}$$

since we are *choosing* the position for the letters/digits from seven positions. Putting it altogether, we get our solution:

$$\binom{7}{4}26^3 \cdot 10^4 = 35 \cdot 26^3 \cdot 10^4$$

4 *Methods of Sampling (time permitting)

If we have n objects and want to sample k of them, how many ways are there to arrange:

4.1 an *ordered* set

Without replacement:

Intuition Imagine an urn, with n uniquely coloured cubes. We pull out k one at a time in sequence.

Solution

$$n \cdot (n-1) \cdots (n-k+1) = \frac{n \cdot (n-1) \cdots (n-k+1) \cdot (n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}$$

With replacement:

Intuition This is simple: If you put the cube back into the urn, you will always have n choices for your k draws. (More interestingly: Is there any connection between this and our discussion about subsets from before?)

Solution n^k

4.2 an UNordered set

Without replacement:

Intuition This is the definition of the binomial coefficient. From n objects we are choosing k objects in a group, and thus not choosing n - k. Reach into the bag, and grab a handful of cubes, their order will be scrambled.

Solution

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

With replacement:

Intuition This idea goes beyond the scope of this course, however with a little thought it is not so bad. It has applications in the non-parametric bootstrap, and particle physics (to name a few). One way to visualize this is the *ball and stick model*, so I created an example below to help illustrate.

Solution

$$\binom{n+k-1}{k}$$

4.2.1 Pizza!

Suppose you have 5 children, and 10 pieces of pizza, how many ways are there to divide up the pizza? (Older siblings can be cruel, so one child could potentially get all 10 slices)

Let's break it down first to a simpler example and see if that helps (often a useful idea in math).

Suppose there are 3 children, and 4 slices of pizza.

Below is a diagram illustrating the separation of the 3 children, by 3-1=2 sticks:

 $\bigcirc \parallel \bigcirc \parallel \bigcirc$

If child 1 gets 2 slices, and the other two get one slice, we have:

 $\bigtriangleup \bigtriangleup \parallel \bigtriangleup \parallel \bigtriangleup$

If child 1 gets 3 slices, child 2 gets 1, child 3 gets none:

 $\triangle \triangle \triangle \parallel \triangle \parallel$

One more for good measure: This time, child 1 has 3, child 2 gets none, child 3 has 1:

 $\bigtriangleup \bigtriangleup \bigtriangleup \Downarrow \parallel \parallel \bigtriangleup$

Let me point out the pattern if you haven't noticed it yet:

There are 3 "objects", in this case the children. And we draw 4 times, which is represented by the pizza going to a random one of the 3 children each time. It doesn't matter when which child gets which slice, all that matters is the final result, the amount of pizza each person gets.

Let's look at our diagrams above now: We have 6 objects (triangles and dividers), and want to *choose* a way to arrange the 2 sticks and 4 slices. So we would expect: $\binom{6}{2} = 15$. In fact, this agrees with our formula: If we have n = 3 children and k = 4 slices, we would then have n - 1 = 2 sticks, giving us

$$\binom{3-1+4}{4} = 15$$

Thus to answer the original question:

$$\binom{5+10-1}{10} = 1001$$

For further reading, consult **Proposition 6.2** in your text which corresponds to: *Distinctive Non-negative Integer-valued vectors* (x_1, x_2, \ldots, x_k) *satisfying* $\sum_{i=1}^k x_i = n$ Be very careful though, although this definition is *closely related in its idea*, it will give you a different answer as the *n* and *k* are switched.

Let's revisit the simpler example and see how. By **Prop. 6.2**, we have the children represented as $x_1 + x_2 + x_3 = 4$, where x_i represents how many slices child i had. n = 4 slices here which gives us the formula:

$$\binom{4+3-1}{3-1} = \binom{6}{2} = 15$$

as required. It of course looks the same, but we defined our n and k differently.

Which again reinforces my original point: There are many ways to solve any given problem, as long as your intuition and mathematics are in sync (positively) you can solve any problem without having to memorize and regurgitate.