Tutorial 6 Suggested Solutions

Probability

Oct. 28

Chapter 4 Example 4b page 123

A product that is sold seasonally yields a net profit of b dollars for each unit sold and a net loss of l dollars for each unit left unsold when the season ends. The number of units of the product that are ordered at a specific department store during any season is a random variable having probability mass function $p(i)$, $i > 0$. If the store must stock this product in advance, determine the number of units the store should stock so as to maximize its expected profit.

Solution:

Let X denote the number of units ordered. If s units are stocked, then the profit, $P(s)$, is:

$P(s) = bX - (s - X)l$ if $X \leq s$

$P(s) = sb$ if $X > s$

Since if there are at most s units ordered the profit is the number of units ordered times the net profit for each unit, but we must subtract the difference in the number of units in stock and the number ordered times the net loss for each unit left unsold. If there are more units ordered than those in stock, then profit is just the number of units in stock times the net profit for each unit.

$E(P(s)) = \sum_{x=0}^{s}\{[bx - (s - x)l] \times p(x)\} + \sum_{x=s+1}^{\infty}\{sb \times p(x)\}$

$E(P(s)) = (b + l)\sum_{x=0}^{s} xp(x) - sl\sum_{x=0}^{s} p(x) + sb[1 - \sum_{x=0}^{s} p(x)]$

$E(P(s)) = (b + l)\sum_{x=0}^{s} xp(x) - (b + l)s\sum_{x=0}^{s} p(x) + sb$

$E(P(s)) = (b + l)\sum_{x=0}^{s}\{(x - s) \times p(x)\} + sb$

To determine the optimum value of s, let us investigate what happens to the profit when we increase s by 1 unit. By substitution, we see that the expected profit in this case is given by

$E(P(s + 1)) = (b + l)\sum_{x=0}^{s+1}\{(x - (s + 1)) \times p(x)\} + (s + 1)b$

$E(P(s + 1)) = (b + l)\sum_{x=0}^{s}\{(x - s - 1) \times p(x)\} + (s + 1)b$

$E(P(s + 1)) = E(P(s)) - (b + l)\sum_{x=0}^{s} p(x) + b$

Thus, stocking $s + l$ units will be better than stocking s units whenever

$$\sum_{x=0}^{s} p(x) < \frac{b}{b+l}$$

Because the left-hand side is increasing in s while the right-hand side is constant, the inequality will be satisfied for all values of $s \leq s*$, where $s*$ is the largest value of s satisfying the equation. Since:

$$E[P(0)] < .... < E[P(s*)] < E[P(s*+1)] > E[P(s*+2)] > ....$$

it follows that stocking $\boldsymbol{s*+1}$ items will lead to a maximum expected profit.

Chapter 4 Example 6f (part a) page 130

A communication system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function. For what values of p is a 5-component system more likely to operate effectively than a 3-component system?

Solution:

Because the number of functioning components is a binomial random variable with parameters (n, p), it follows that the probability that a 5-component system will be effective is:

$\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + \binom{5}{5}p^5(1-p)^0 = 10p^3(1-p)^2 + 5p^4(1-p) + p^5$

whereas the corresponding probability for a 3-component system is

$\binom{3}{2}p^2(1-p)^1 + \binom{3}{3}p^3 = 3p^2(1-p) + p^3$

The 5-component system is better if:

$10p^3(1-p)^2 + 5p^4(1-p) + p^5$ ¿ $3p^2(1-p) + p^3$

which reduces to:

$3(p-l)^2(2p-1) > 0$

which reduces to:

$(2p-1) > 0$

or $\boldsymbol{p > \frac{1}{2}}$

Chapter 4 Problem 4.14 page 164

Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find

$PX = i$, $i = 0, 1, 2, 3, 4$, and $E(X)$.

X=0

In this case player 1 never wins. That is, player 1 loses to player 2. If we construct the sample space, we would see that $P(X = 0) = \frac{1}{2}$ because exactly one-half of the 5! permutations have the first number (player 1) greater than the second number (player 2).

$$\boldsymbol{P(X = 0) = \frac{\frac{1}{2!}5!}{5!} = \frac{1}{2}}$$

X=1

Here player 1 wins exactly 1 game if player 3 has a larger number than player 1, but player 1 has a larger number than player 2. The number of ways this can happen is the same as the number of ways that player 2 loses to player 1 and player 3. Thus, $P(X = 1) = P(Y_2 < Y_1 < Y_3)$ where Y_i denotes the number given to player i. When $i \neq j \neq k$, there are 3! ways to arrange the inequality $Y_i < Y_j < Y_k$. Exactly one of these arrangements gives us the inequality $Y_2 < Y_1 < Y_3$. Therefore,

$$\boldsymbol{P(X = 1) = \frac{\frac{1}{3!}5!}{5!} = \frac{1}{6}}$$

X=2

Now consider the case that player 1 wins exactly 2 games. This means $Y_1 > Y_2$ and $Y_1 > Y_3$ but $Y_1 < Y_4$. The easiest way to do this is to consider all of the possible ways that this can happen:

$S_{\text{Player 1 wins 2 games}} = \{(Y_1, Y_2, Y_3, Y_4)\}$

$S_{\text{Player 1 wins 2 games}} = \{(3, 1, 2, 4), (3, 1, 2, 5), (3, 2, 1, 4), (3, 2, 1, 5), (4, 1, 2, 5), (4, 2, 1, 5), (4, 1, 3, 5), (4, 3, 1, 5), (4, 2, 3, 5), (4, 3, 2, 5)\}$

Thus, we find:

$$\boldsymbol{P(X = 2) = \frac{10}{5!} = \frac{1}{12}}$$

X=3

Now consider the case that player 1 wins exactly 3 games. This means $Y_1 > Y_2$, $Y_1 > Y_3$, and $Y_1 > Y_4$, but $Y_1 < Y_5$. Thus, logically $Y_1 = 4$, and $Y_5 = 5$, and vary the others, giving 3! arrangements. Or, we can just list out the different combinations.

$S_{\text{Player 1 wins 3 games}} = \{(Y_1, Y_2, Y_3, Y_4, Y_5)\}$

$S_{\text{Player 1 wins 3 games}} = \{(4, 1, 2, 3, 5), (4, 2, 3, 1, 5), (4, 3, 2, 1, 5), (4, 1, 3, 2, 5), (4, 2, 1, 3, 5), (4, 3, 1, 2, 5)\}$

$$\boldsymbol{P(X = 3) = \frac{3!}{5!} = \frac{1}{20}}$$

X=4

For player 1 to win all 4 games. This means $Y_1 > Y_2$, $Y_1 > Y_3$, $Y_1 > Y_4$, and $Y_1 > Y_5$. Thus, $Y_1 = 5$ and the remaining variables can be anything less than 5. Thus there are 4! possible ways to arrange Y_2, Y_3, Y_4, and Y_5.

$S_{\text{Player 1 wins 4 games}} = \{(Y_1, Y_2, Y_3, Y_4, Y_5)\}$

$S_{\text{Player 1 wins 4 games}} = \{(5, *, *, *, *)\}$

$$\boldsymbol{P(X=4) = \frac{4!}{5!} = \frac{1}{5}}$$

Now we will solve for the expected value:

$E(X) = \sum_{i=0}^{5} iP(X = i)$

$E(X) = 0(\frac{1}{2}) + 1(\frac{1}{6}) + 2(\frac{1}{12}) + 3(\frac{1}{20}) + 4(\frac{1}{5})$

$\boldsymbol{E(X) = 1.28333333}$