Tutorial 4 Suggested Solutions

Probability

Oct. 7

Suggested Problem 1

You have 12 red balls and 12 blue balls to distribute among 3 urns. You can distribute them among the urns any way that you want (the urns do not have to have equal numbers of balls in them). How would you distribute them to maximize the possibility that a random ball drawn from a randomly chosen urn is blue, and what is that probability?

Solution:
54/66 = 82%
By putting a single blue ball in each of two of the urns, and the other 22 balls in the 3rd urn. Any other break down yields smaller probabilities.

Suggested Problem 2

A man is at a park where he's just had to deal with his complaining daughter. He sits on a bench beside a woman and strikes up a conversation:

a. Man: Have you any kids? Woman: Yes, 2.
   What's the probability that she has 2 girls?

b. Man: Any girls? Woman: Yes.
   What's the probability that she has 2 girls? 1/3

c. Just then a little girl runs up to the woman saying "Mommy! Mommy!"
   What's the probability that that girl's sibling is also a girl?

Solution:

a. Since the woman has 2 children the sample space is: $S = \{BB, BG, GB, GG\}$, there is a **1/4** probability that she has two girls.

b. Since the woman has 2 children (from a), and at least one girl the sample space is: $S = \{BG, GB, GG\}$, there is a **1/3** probability that she has two girls using the given information.

c. The children's genders are independent. Thus we are now asking what is the probability that the little girl's sibling is also a girl. Well the sample space for the girls other sibling is: $S = \{B, G\}$. There is a **1/2** probability that the girl's sibling is also a girl, and thus a **1/2** probablity that the woman has two girls.

Page 68 - Example 3i

An urn contains two type A coins and one type B coin. When a type A coin is flipped, it comes up heads with probability 1/4, whereas when a type B coin is flipped, it comes up heads with probability 3/4. A coin is randomly chosen from the urn and flipped. Given that the flip landed on heads, what is the probability that it was a type A coin?

Solution:
Let A be the event that a type A coin was flipped, and let $B = A^c$ be the event that a type B coin was flipped. We want $P(A|heads)$, where heads is the event that the flip landed on heads. We know $P(heads|A)$ =1/4, $P(heads|B)$ =3/4, $P(A)$ =2/3 (since there are 2 A coins, and 1 B coin), and $P(B)$ =1/3. Since we want $P(A|heads)$, and we know $P(heads|A)$ we can use Bayes' formula:

$$P(A|heads) = \frac{P(heads|A)P(A)}{P(heads|A)P(A) + P(heads|B)P(B)}$$

$$P(A|heads) = \frac{(\frac{1}{4})(\frac{2}{3})}{(\frac{1}{4})(\frac{2}{3}) + (\frac{3}{4})(\frac{1}{3})} = \mathbf{\frac{2}{5}}$$

Page 70 - Example 3k

A plane is missing, and it is presumed that it was equally likely to have gone down in any of 3 possible regions. Let 1-β_i, $i = 1, 2, 3$, denote the probability that the plane will be found upon a search of the i^{th} region when the plane is, in fact, in that region. (The constants β_i are called overlook probabilities, because they represent the probability of overlooking the plane; they are generally attributable to the geographical and environmental conditions of the regions.) What is the conditional probability that the plane is in the ith region given that a search of region 1 is unsuccessful?

Solution: Let R_i for $i = 1, 2, 3$, be the event that the plane is in region i, and let E be the event that a search of region 1 is unsuccessful. Thus we wish to find $P(R_i|E)$ for $i = 1, 2, 3$.

For Region 1:

$$P(R_1|E) = \frac{P(E|R_1)P(R_1)}{P(E|R_1)P(R_1) + P(E|R_2)P(R_2) + P(E|R_3)P(R_3)} \qquad \text{(by Bayes' formula)}$$

From the question, we can see that $P(E|R_1) = \beta_1$, $P(E|R_2) = P(E|R_3) = 1$ (since R_2, and R_3 imply the plane is not in region 1, the probability of an unsuccessful search in region 1 is guaranteed). And $P(R_i) = \frac{1}{3}$, since there's an equal probability of the plane landing in any of the 3 regions.

$$P(R_1|E) = \frac{(\beta_1)\frac{1}{3}}{\beta_1\frac{1}{3} + (1)\frac{1}{3} + (1)\frac{1}{3}} = \boldsymbol{\frac{\beta_1}{\beta_1 + 2}}$$

For Region 2:

$$P(R_2|E) = \frac{P(E|R_2)P(R_2)}{P(E|R_1)P(R_1) + P(E|R_2)P(R_2) + P(E|R_3)P(R_3)} \qquad \text{(by Bayes' formula)}$$

Again, $P(E|R_1) = \beta_1$, $P(E|R_2) = P(E|R_3) = 1$ and $P(R_i) = \frac{1}{3}$.

$$P(R_2|E) = \frac{(1)\frac{1}{3}}{\beta_1\frac{1}{3} + (1)\frac{1}{3} + (1)\frac{1}{3}} = \mathbf{\frac{1}{\beta_1+2}}$$

For Region 3:

Similar to region 2: $P(R_3|E) = \mathbf{\frac{1}{\beta_1+2}}$

Page 73 - Example 3n

A bin contains 3 different types of disposable flashlights. The probability that a type 1 flashlight will give over 100 hours of use is 0.7, with the corresponding probabilities for type 2 and type 3 flashlights being 0.4 and 0.3, respectively. Suppose that 20% of the flashlights in the bin are type 1, 30% are type 2, and 50% are type 3.

a. What is the probability that a randomly chosen flashlight will give more than 100 hours of use?

b. Given that a flashlight lasted over 100 hours, what is the conditional probability that it was a type j flashlight, j = 1, 2, 3?

Solution:

a. Let A denote the event that a flashlight lasts more than 100 hours. We want $P(A)$. We can partition this into the cases of which flashlight is used (denoted by F_j, for j =1,2,3):

$$P(A) = P(A \cap F_1) + P(A \cap F_2) + P(A \cap F_3) \qquad (1)$$

By rearranging the conditional formulas $P(A|F_i) = \frac{P(A \cap F_j)}{P(F_j)}$ we can fill in $P(A \cap F_j)$ for $j = 1, 2, 3$ in (1).

$$P(A) = P(A|F_1)P(F_1) + P(A|F_2)P(F_2) + P(A|F_3)P(F_3)$$

$$P(A) = (0.7)(0.2) + (0.4)(0.3) + (0.3)(0.5) = \mathbf{0.41}$$

b. To find $P(F_j|A)$, we open up the conditional probability formula. And use $P(A) = 0.41$ from part *(a)*.

$$P(F_1|A) = \frac{P(F_1A)}{P(A)} = \frac{P(A|F_1)P(F_1)}{0.41} = \frac{0.7 \times 0.2}{0.41} = \mathbf{\frac{14}{41}}$$
$$P(F_2|A) = \frac{P(F_2A)}{P(A)} = \frac{P(A|F_2)P(F_2)}{0.41} = \frac{0.4 \times 0.3}{0.41} = \mathbf{\frac{12}{41}}$$
$$P(F_3|A) = \frac{P(F_3A)}{P(A)} = \frac{P(A|F_3)P(F_3)}{0.41} = \frac{0.3 \times 0.5}{0.41} = \mathbf{\frac{15}{41}}$$

Page 80 - Example 4i

Suppose there are n types of coupons and that each new coupon collected is independent of previous selections, a type i coupon with probability p_i, $\sum_{i=1}^{n} pi = 1$. Suppose k coupons are to be collected. If A_i is the event that there is at least one type i coupon among those collected, then, for $i \neq j$, find:

a. $P(A_i)$

b. $P(A_i \cup A_j)$

c. $P(A_i|A_j)$

Solution:

a. We know that k independent coupons are selected, this the probability that each selected coupon is not of type i is $1 - p_i$. Thus the probability that none of the selected coupons are of type i is $(1 - p_i)^k$.

$$P(A_i) = 1 - P(A_i^c) = 1 - P(\text{no coupon is type } i) = \mathbf{1 - (1 - \boldsymbol{p_i})^{\boldsymbol{k}}}$$

b. Similar to part (a.) we know the probability that each selected coupon is not of type i nor j is $1 - p_i - p_j$. Thus, the probability that none of the k independently selected coupons are of type i or j is $(1 - p_i - p_j)^k$.

$$P(A_i \cup A_j) = 1 - P((A_i \cup A_j)^c) = 1 - P(\text{no coupon is type } i \text{ or } j) = \mathbf{1 - (1 - \boldsymbol{p_i} - \boldsymbol{p_j})^{\boldsymbol{k}}}$$

c. We can open up the conditional probability formula, but we will need to find the $P(A_iA_j)$ first. To do this we will rearrange the union in part (b.) in conjunction with $P(A_i)$ from part (a.):

$$P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_iA_j)$$

$$P(A_iA_j) = P(A_i) + P(A_j) - P(A_i \cup A_j) \qquad \textit{(Using part (a.) and (b.))}$$

$$P(A_iA_j) = 1 - (1 - p_i)^k + 1 - (1 - p_j)^k - (1 - (1 - p_i - p_j)^k)$$

$$P(A_iA_j) = 1 - (1 - p_i)^k - (1 - p_j)^k + (1 - p_i - p_j)^k$$

Now we have $P(A_iA_j)$, so we can expand the conditional probability formula to solve for $P(A_i|A_j)$:

$$P(A_i|A_j) = \frac{P(A_iA_j)}{P(A_j)} \qquad \textit{(Using } P(A_iA_j) \textit{ and part (a.))}$$

$$P(A_i|A_j) = \frac{\mathbf{1 - (1 - \boldsymbol{p_i})^{\boldsymbol{k}} - (1 - \boldsymbol{p_j})^{\boldsymbol{k}} + (1 - \boldsymbol{p_i} - \boldsymbol{p_j})^{\boldsymbol{k}}}}{\mathbf{1 - (1 - \boldsymbol{p_j})^{\boldsymbol{k}}}}$$