# Tutorial 5 Suggested Solutions 

Probability

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## Chapter 3 Example 5b page 91

A female chimp has given birth. It is not certain, however, which of two male chimps is the father. Before any genetic analysis has been performed, it is felt that the probability that male number 1 is the father is $p$ and the probability that male number 2 is the father is $1-p$. DNA obtained from the mother, male number 1 , and male number 2 indicate that, on one specific location of the genome, the mother has the gene pair $(A, A)$, male number 1 has the gene pair $(a, a)$, and male number 2 has the gene pair $(A, a)$. If a DNA test shows that the baby chimp has the gene pair $(A, a)$, what is the probability that male number 1 is the father?

## Solution:

We know that the mother has the gene pair $(A, A)$, male number 1 has the gene pair $(a, a)$, and male number 2 has the gene pair $(A, a)$. These gene pairs are given information, so we can condition on them.

Let $M_{i}$ be the event that male number $i, i=1,2$, is the father, and let $B_{(A, a)}$ be the event that the baby chimp has the gene pair $(A, a)$. Then $P\left(M_{1} \mid B_{(A, a)}\right)$ is:

$$
\begin{gathered}
P\left(M_{1} \mid B_{(A, a)}\right)=\frac{P\left(M_{1} B_{(A, a)}\right)}{P\left(B_{(A, a)}\right)} \quad \text { From the conditional probability formula } \\
P\left(M_{1} \mid B_{(A, a)}\right)=\frac{\left.P_{(A, a)} \mid M_{1}\right) P\left(M_{1}\right)}{P\left(B_{(A, a)} \mid M_{1}\right) P\left(M_{1}\right)+P\left(B_{(A, a)} \mid M_{2}\right) P\left(M_{2}\right)}
\end{gathered}
$$

We know $P\left(B_{(A, a)} \mid M_{1}\right)=1$, since the baby will get one gene from the mother and one from the father. And the probability of getting the gene combination $(A, a)$ is certain if the mom has $(A, A)$ and the dad has $(a, a)$. We know $P\left(B_{(A, a)} \mid M_{2}\right)=1 / 1$, since the probability of getting the gene combination $(A, a)$ is certain if the mom has $(A, A)$ and $1 / 2$ if the dad has $(A, a)$.

From the question we also know $P\left(M_{1}\right)=p$ and $P\left(M_{2}\right)=1-p$. So:

$$
\begin{aligned}
& P\left(M_{1} \mid B_{(A, a)}\right)=\frac{(1) \times p}{(1) \times p+(1 / 2) \times(1-p)} \\
& P\left(M_{1} \mid B_{(A, a)}\right)=\frac{2 p}{1+p}
\end{aligned}
$$

Since $\frac{2 p}{1+p}>p$ when $p<1$, the information that the baby's gene pair is $(A, a)$ increases the probability that male number 1 is the father. This result is intuitive because it is more likely that the baby would have gene pair $(A, a)$ if $M_{1}$ is true than if $M_{2}$ is true (the respective conditional probabilities being 1 and $1 / 2$ ).

## Chapter 3 Example 5c page 92

Independent trials, each resulting in a success with probability $p$ or a failure with probability $q=1-p$, are performed. We are interested in computing the probability that a run of $n$ consecutive successes occurs before a run of $m$ consecutive failures.

## Solution:

Let $E$ be the event that a run of n consecutive successes occurs before a run of $m$ consecutive failures. To obtain $P(E)$, we start by conditioning on the outcome of the first trial. That is, letting $H$ denote the event that the first trial results in a success. Thus:

$$
P(E)=P(E H)+P\left(E H^{c}\right)=P(H) P(E \mid H)+P\left(H^{c}\right) P\left(E \mid H^{c}\right)
$$

But we know from the question $P(H)=p$ and $P\left(H^{c}\right)=1-p=q$

$$
P(E)=p P(E \mid H)+q P\left(E \mid H^{c}\right)
$$

Now, given that the first trial was successful, one way we can get a run of $n$ successes before a run of $m$ failures would be to have the next $n-1$ trials all result in successes. So, let us condition on whether or not that occurs. That is, letting $F$ be the event that trials 2 through $n$ all are successes.
$P(E \mid H)=P(E \mid F H) P(F \mid H)+P\left(E \mid F^{c} H\right) P\left(F^{c} \mid H\right)$
If $F H$ occurs then $E$ must occur thus $P(E \mid F H)=1$. If the event $F^{c} H$ occurs, then the first trial would result in a success, but there would be a failure some time during the next $n-1$ trials. However, when this failure occurs, it would wipe out all of the previous successes, and the situation would be exactly as if we started out with a failure. So:

$$
P\left(E \mid F^{c} H\right)=P\left(E \mid H^{c}\right)
$$

Because the independence of trials implies that $F a n d H$ are independent, and because $P(F)=p^{(n-1)}$, then:

$$
P(E \mid H)=p^{(n-1)}+\left(1-p^{(n-1)}\right) P\left(E \mid H^{c}\right)
$$

Similarily
$P\left(E \mid H^{c}\right)=P\left(E \mid G H^{c}\right) P\left(G \mid H^{c}\right)+P\left(E \mid G^{c} H^{c}\right) P\left(G^{c} \mid H^{c}\right)$
Now, $G H^{c}$ is the event that the first $m$ trials all result in failures, so $P\left(E \mid G H^{c}\right)=0$. Also, if $G^{c} H^{c}$ occurs, then the first trial is a failure, but there is at least one success in the next $m-1$ trials. Hence, since this success wipes out all previous failures, we see that:
$P\left(E \mid G^{c} H^{c}\right)=P(E \mid H)$
We also can notice that $P\left(G^{c} \mid H^{c}\right)=P\left(G^{c}\right)=1-q^{m-1}$, thus we can expand:

$$
P\left(E \mid H^{c}\right)=\left(1-q^{m-1}\right) P(E \mid H)
$$

So now we can solve for $P(E \mid H)$ and $P\left(E \mid H^{c}\right)$ :

$$
\begin{aligned}
& P(E \mid H)=\frac{p^{n-1}}{p^{n-1}+q^{m-1}-p^{n-1} q^{m-1}} \\
& P\left(E \mid H^{c}\right)=\frac{\left(1-q^{m-1}\right) p^{n-1}}{p^{n-1}+q^{m-1}-p^{n-1} q^{m-1}}
\end{aligned}
$$

Using $P(E \mid H)$ and $P\left(E \mid H^{c}\right)$, we can sub into $P(E)=p P(E \mid H)+q P\left(E \mid H^{c}\right)$ :

$$
\begin{aligned}
& P(E)=\frac{p^{n}+q p^{n-1}\left(1-q^{m-1}\right)}{p^{n-1}+q^{m-1}-p^{n-1} q^{m-1}} \\
& P(E)=\frac{\left.\boldsymbol{p}^{\boldsymbol{n}-\mathbf{1}} \mathbf{(} \mathbf{-} \boldsymbol{q}^{\boldsymbol{m}}\right)}{\boldsymbol{p}^{\boldsymbol{n}-\mathbf{1}}+\boldsymbol{q}^{\boldsymbol{m}-\mathbf{1}}-\boldsymbol{p}^{\boldsymbol{n - 1}} \boldsymbol{q}^{\boldsymbol{m - 1}}}
\end{aligned}
$$

## Chapter 4 Example 1b page 113

A life insurance agent has 2 elderly clients, each of whom has a life insurance policy that pays $\$ 100,000$ upon death. Let $Y$ be the event that the younger one dies in the following year, and let $O$ be the event that the older one dies in the following year. Assume that $Y$ and $O$ are independent, with respective probabilities $P(Y)=.05$ and $P(O)=.10$. If $X$ denotes the total amount of money (in units of $\$ 100,000$ ) that will be paid out this year to any of these clients' beneficiaries, what are the possible values of $X$ and their associated probabilities.

## Solution:

$X$ is a random variable that takes on one of the possible values $0,1,2$. Since either 0 or 1 or both people could die.

Case when $X=0$ :
$P(X=0)=P($ neither dies $)=P\left(Y^{c} O^{c}\right)$

Since $Y$ and $O$ are independent then $P\left(Y^{c} O^{c}\right)=P\left(Y^{c}\right) P\left(O^{c}\right)$, thus:
$P(X=0)=P\left(Y^{c}\right) P\left(O^{c}\right)=(1-0.05)(1-0.1)=\mathbf{0 . 8 5 5}$
Case when $X=1$ :
$P(X=1)=P($ exactly one dies $)$
$P(X=1)=P\left(Y^{c} O \cup Y O^{c}\right)$
$P(X=1)=P\left(Y^{c} O\right)+P\left(Y O^{c}\right)$ since events $Y^{c} O$ and $Y O^{c}$ are mutually exclusive.
$P(X=1)=P\left(Y^{c}\right) P(O)+P(Y) P\left(O^{c}\right) \quad$ from indepence
$P(X=1)=(1-0.05)(0.1)+(0.05)(1-0.1)=\mathbf{0 . 1 4}$

## Case when $X=2$ :

$$
\begin{aligned}
& P(X=2)=P(\text { both die })=P(Y O) \\
& P(X=2)=P(Y) P(O) \quad \text { from indepence }
\end{aligned}
$$

$$
P(X=2)=(0.05)(0.1)=\mathbf{0 . 0 0 5}
$$

