Tutorial 8

Ch. 4 and 5

Nov. 11

Summary of the discrete distributions:

X	*X Counts*	*p(x)*	*Values of X*	*E(x)*	*V(x)*
Binomial	Number of sucesses in n fixed trials	$\binom{n}{x} p^x (1-p)^{n-x}$	$x = 0,1,...,n$	np	$np(1-p)$
Poisson	Number of arrivals in a fixed time period	$\frac{e^{-\lambda}\lambda^x}{x!}$	$x = 0,1,2,...$	λ	λ
Geometric	Number of trials up through 1st success	$(1-p)^{x-1}p$	$x = 1,2,3,...$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative Binomial	Number of trials up through kth success	$\binom{x-1}{k-1}(1-p)^{x-k}p^k$	$x = k, k+1,...$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$
Hyper - geometric	Number of marked individuals in sample taken without replacement	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$	$\max(0, M+n-N) \le x \le \min(M,n)$	$n * \frac{M}{N}$	$\frac{nM(N-M)(N-n)}{N^2(N-1)}$

Chapter 4 Example 9c page 157

Find the expected value of the sum obtained when n fair dice are rolled.

Solution:

Let X represent to total sum. We can re-write X as: $\sum_{i=1}^{n} X_i$ where X_i is the upturned value on die i for $i = 1, ..., n$. Because X_i is equally likely to be any of the values from 1 to 6 we know that:

$E(X_i) = \sum_{j=1}^{6} j\frac{1}{6}$

$E(X_i) = 3.5$

So if we wish to find $E(X)$ we are finding $E(\sum_{i=1}^{n} X_i)$.

$E(X) = E(\sum_{i=1}^{n} X_i)$

$E(X) = \sum_{i=1}^{n} E(X_i)$

$E(X) = \sum_{i=1}^{n} 3.5$

$E(X) = 3.5n$

Chapter 5 Problem 5.1 page 212

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & otherwise \end{cases}$$

a. What is the value of c?

b. What is the cumulative distribution function?

Solution:

Part a.

We know that the area under a density curve should be 1. Thus: $\int_{-\infty}^{\infty} f(x)dx = 1$.

$1 = \int_{-\infty}^{\infty} f(x)dx$
$1 = \int_{-\infty}^{-1} 0dx + \int_{-1}^{-1} c(1-x^2)dx + \int_{-1}^{\infty} 0dx$
$1 = \int_{-1}^{-1} c(1-x^2)dx$
$1 = c\int_{-1}^{-1} (1-x^2)dx$
$1 = c[x - \frac{x^3}{3}]|_{x=-1}^{x=1}$
$1 = c[1 - \frac{1}{3}] - c[-1 - \frac{-1}{3}]$
$1 = c[\frac{4}{3}]$
$\boldsymbol{c = \frac{3}{4}}$

Part b.

We know that the cumulative density function is defined as $F(x) = P(X \leq x)$. Thus: $F(x) = \int_{-\infty}^{x} f(x)dx$. Which is the are under the density function to the left of x.

$F(x) = \int_{-\infty}^{x} f(t)dt$

For $-\infty < x < -1$ the area below the curve to the left of x is just 0:
$F(x) = \int_{-\infty}^{x} 0dt$
$F(x) = 0$

For $-1 < x < 1$ the area below the curve to the left of x is:
$F(x) = \int_{-\infty}^{-1} 0dt + \int_{-1}^{x} 0.75(1-t^2)dt$
$F(x) = 0.75\int_{-1}^{t} (1-t^2)dt$
$F(x) = 0.75(t - \frac{t^3}{3})|_{t=-1}^{t=x}$
$F(x) = 0.75(x - \frac{x^3}{3} - (-1 + 1/3))$
$F(x) = 0.75x - \frac{x^3}{4} + \frac{1}{2}$

For $1 < x < \infty$ the area below the curve to the left of x is:
$F(x) = \int_{-\infty}^{-1} 0dt + \int_{-1}^{1} 0.75(1-t^2)dt + \int_{1}^{t} 0dt$
$F(x) = 1$

Thus:

$$F(x) = \begin{cases} 0 & x < -1 \\ 0.75x - \frac{x^3}{4} + \frac{1}{2} & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Suggested Problem

Find the cumulative distribution function of the following density:

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 1 & 1 \leq x < 1.5 \\ 0 & otherwise \end{cases}$$

Solution:

We know that the cumulative density function is defined as $F(x) = P(X \leq x)$. Thus: $F(x) = \int_{-\infty}^{x} f(x)dx$. Which is the are under the density function to the left of x.

$F(x) = \int_{-\infty}^{x} f(t)dt$

For $-\infty < x < 0$ the area below the curve to the left of x is just 0:
$F(x) = \int_{-\infty}^{x} 0dt$
$F(x) = 0$

For $0 < x < 1$ the area below the curve to the left of x is:
$F(x) = \int_{-\infty}^{0} 0dt + \int_{0}^{x} tdt$
$F(x) = \int_{0}^{t} tdt$
$F(x) = \frac{t^2}{2}|_{t=0}^{t=x}$
$F(x) = \frac{x^2}{2} - \frac{0}{2}$
$F(x) = \frac{x^2}{2}$

For $1 < x < 1$ the area below the curve to the left of x is:
$F(x) = \int_{-\infty}^{0} 0dt + \int_{0}^{1} tdt + \int_{1}^{x} 1dt$
$F(x) = \frac{t^2}{2}|_{t=0}^{t=1} + t|_{t=1}^{t=x}$
$F(x) = \frac{1}{2} + (x - 1)$
$F(x) = x - \frac{1}{2}$

For $1.5 < x < \infty$ the area below the curve to the left of x is:
$F(x) = \int_{-\infty}^{0} 0dt + \int_{0}^{1} tdt + \int_{1}^{1.5} 1dt \int_{1.5}^{x} 0dt$
$F(x) = \frac{t^2}{2}|_{t=0}^{t=1} + t|_{t=1}^{t=1.5}$
$F(x) = \frac{1}{2} + (1.5 - 1)$
$F(x) = 1$

Thus:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \leq x \leq 1 \\ x - \frac{1}{2} & 1 < x \leq 1.5 \\ 1 & x > 1.5 \end{cases}$$