Tutorial 11

Ch. 6

Dec. 2

Chapter 6 Example 1d page 225

Consider a circle of radius R, and suppose that a point within the circle is randomly chosen in such a manner that all regions within the circle of equal area are equally likely to contain the point. (In other words, the point is uniformly distributed within the circle.) If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen (Figure 6.1), then, since (X, Y) is equally likely to be near each point in the circle, it follows that the joint density function of X and Y is given by:

$$f(x) = \begin{cases} c & x^2 + y^2 \leq R \\ 0 & x^2 + y^2 > R \end{cases}$$

for some value of c.

a. Find c.

b. Find the marginal density functions of X and Y.

c. Compute the probability that D, the distance from the origin of the point selected, is less than or equal to a.

d. Find $E[D]$.

Solution:

Part a.

We know $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$ so it follows that:

$\int \int_{x^2+y^2 \leq R} c dx dy = 1$

$c \int \int_{x^2+y^2 \leq R} dx dy = 1$

but $\int \int_{x^2+y^2 \leq R} dx dy$ is the area of the circle with radius R which is πR^2, thus:

$c\pi R^2 = 1$

$c = \frac{1}{\pi R^2}$

Thus the joint density is:

$$f(x) = \begin{cases} \frac{1}{\pi R^2} & x^2 + y^2 \leq R \\ 0 & x^2 + y^2 > R \end{cases}$$

Part b.

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2} dy \qquad$ for $x^2 \leq R^2$

$f_X(x) = \frac{1}{\pi R^2} y|_{y=-\sqrt{R^2-x^2}}^{y=\sqrt{R^2-x^2}} \qquad$ for $x^2 \leq R^2$

$f_X(x) = \frac{1}{\pi R^2}(\sqrt{R^2-x^2} - (-\sqrt{R^2-x^2})) \qquad$ for $x^2 \leq R^2$

$f_X(x) = \frac{2\sqrt{R^2-x^2}}{\pi R^2} \qquad$ for $x^2 \leq R^2$

and by symmetry: $f_Y(y) = \frac{2\sqrt{R^2-y^2}}{\pi R^2} \qquad$ for $y^2 \leq R^2$

Part c.

The distribution for $D = \sqrt{X^2+Y^2}$ the distance from the origin, is obtained as follows, for $0 \leq a \leq R$:

$F_D(a) = P(\sqrt{X^2+Y^2} \leq a)$

$F_D(a) = P(X^2+Y^2 \leq a^2)$
$F_D(a) = \int\int_{x^2+y^2 \leq a} \frac{1}{\pi R^2} dxdy$

$F_D(a) = \frac{1}{\pi R^2} \int\int_{x^2+y^2 \leq a} dxdy \qquad$ this is just the area of the circle with radius a.

$F_D(a) = \frac{1}{\pi R^2}[\pi a^2]$

$F_D(a) = \frac{a^2}{R^2}$

Part d.

To find $E[D]$ we use the CDF of D (from part c) and differentiate it to find the pdf. Then use the pdf to find the expected value.

Since $F_D(a) = \frac{a^2}{R^2}$ then the derivative of $F_D(a)$ with respect to a gives us the pdf:

$f_D(a) = \frac{2a}{R^2} \qquad$ for $0 \leq a \leq R$

Finding the expected value:

$E[D] = \int_{-infty}^{\infty} a f_D(a) da$

$E[D] = \int_0^R a \frac{2a}{R^2} da$

$E[D] = \frac{2}{R^2} \int_0^R a^2 da$

$E[D] = \frac{2}{R^2}[\frac{a^3}{3}]|_{a=0}^{a=R}$

$E[D] = \frac{2}{R^2}[\frac{R^3}{3}]$

$E[D] = \frac{2R}{3}$

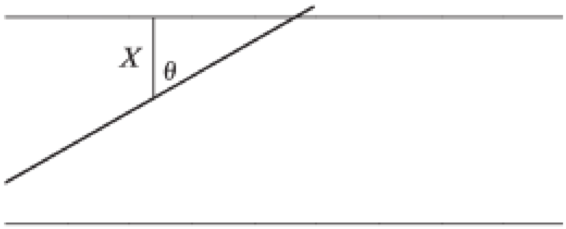
Chapter 6 Example 2d page 231 - Buffon's Needle Problem

A table is ruled with equidistant parallel lines a distance D apart. A needle of length L, where $L \leq D$, is randomly thrown on the table. What is the probability that the needle will intersect one of the lines (the other possibility being that the needle will be completely contained in the strip between two lines)?

Solution:

Let us determine the position of the needle by specifying the distance X from the middle point of the needle to the nearest parallel line and the angle θ between the needle and the projected line of length X. The needle will intersect a line if the hypotenuse of the right triangle in the drawing is less than $L/2$, that is, if:

$\frac{X}{cos(\theta)} < \frac{L}{2}$ OR $X < \frac{L \times cos(\theta)}{2}$



X varies from 0 and $D/2$ and θ varies between 0 and $\pi/2$. We assume that X and θ are independent, and uniformly distributed random variables.

$P(X < \frac{L}{2}cos\theta) = \int\int_{x<\frac{L}{2}cosy} f_X(x) f_\theta(y) dxdy$

$P(X < \frac{L}{2}cos\theta) = \int_0^{\pi/2}\int_0^{L/2cos(y)} \frac{1}{D/2}\frac{1}{pi/2} dxdy$

$P(X < \frac{L}{2}cos\theta) = \frac{4}{\pi D}\int_0^{\pi/2}\int_0^{L/2cos(y)} dxdy$

$P(X < \frac{L}{2}cos\theta) = \frac{4}{\pi D}\int_0^{\pi/2} x|_{x=0}^{x=L/2cos(y)} dy$

$P(X < \frac{L}{2}cos\theta) = \frac{4}{\pi D}\int_0^{\pi/2} \frac{L \times cos(y)}{2} dy$

$P(X < \frac{L}{2}cos\theta) = \frac{2L}{\pi D}\int_0^{\pi/2} cos(y) dy$

$P(X < \frac{L}{2}cos\theta) = \frac{2L}{\pi D}[sin(y)]|_{y=0}^{y=\pi/2}$

$P(X < \frac{L}{2}cos\theta) = \frac{2L}{\pi D}$

Chapter 6 Example 2h page 236

Let X, Y, Z be independent and uniformly distributed over $(0, 1)$. Compute $P(X \geq YZ)$.

Solution:

Since X, Y, Z are independent, then $f_{X,Y,Z}(x, y, z) = f_X(x)f_Y(y)f_Z(z) = 1$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

$P(X \geq YZ) = \int \int \int_{x \geq yz} f_{X,Y,Z}(x, y, z)dxdydz$

$P(X \geq YZ) = \int_0^1 \int_0^1 \int_y z^1 1dxdydz$

$P(X \geq YZ) = \int_0^1 \int_0^1 (1 - yz)dydz$

$P(X \geq YZ) = \int_0^1 (y - y^2 z/2)|_{y=0}^{y=1} dz$

$P(X \geq YZ) = \int_0^1 (1 - z/2)dz$

$P(X \geq YZ) = z - z^2/4|_{z=0}^{z=1} dz$

$P(X \geq YZ) = 1 - 1/4$

$P(X \geq YZ) = 3/4$