Assignment 2 / Due Wednesday July 4

PROBLEM 1

Find the extreme values of $f(x,y,z) = x + y^2z$, subject to the constraints $y^2 + z^2 = 2$ and $z = 2$.

Solution and hint: If $g_1 = y^2 + z^2 - 2$ and $g_2 - 2$, then

$$\begin{cases} \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$$

From the first equation, we have $0 = 1$, which is impossible. Therefore, this system has no solution, so there are no maximum and minimum as well.

PROBLEM 2

Show that the following equations can be solved for u and v as functions of x, y and z, near the point P_0, where $(x,y,z) = (2,0,1)$ and $(u,v) = (1,0)$. Find $\frac{\partial u}{\partial z}$ at this point.

$$\begin{cases} xe^y + uz - \text{cos}v = 2 \\ u\text{cos}v + x^2v - yz^2 = 1 \end{cases}$$

Solution and hint: We take the derivatives of the equations, with respect to u and v.

$$\begin{bmatrix} z & \sin v \\ \cos v & -u\sin v + x^2 \end{bmatrix}$$

At the point P_0, this matrix is equal to

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix} \Rightarrow \det = 4 \neq 0$$

The matrix is invertible, and by the Implicit Function Theorem, we can solve for u and v. To find u_z, take the derivative of both the equations by implicit differentiation, and notice that, u and v, are functions of x, y and z. From the first equation, we have

$$u_z z + u - v_z \sin v = 0 \xrightarrow{\text{at } P_0} u_z + 1 = 0 \Rightarrow u_z = -1$$

Note that in general, we need to take the derivative of both the equations with respect to z, to obtain a system of two equations and two unknowns (u_z and v_z). Solving that system gives us the answer. Here the equations are special, and we didn't need to find the derivative of the second equation.

PROBLEM 3

Find the maximum and minimum values for the curvature of the ellipse $x = a\text{cos}t$, $y = b\text{sin}t$, where $a > b > 0$.

Solution and hint:

As a space curve, we can write c as $c(t) = (a\cos t, b\sin t, 0)$, and we have

$$\begin{aligned} c'(t) &= (-a\text{sin}t, b\text{cos}t, 0) \\ c''(t) &= (-a\text{cos}t, -b\text{sin}t, 0) \\ c'(t) \times c''(t) &= (0,0,ab) \\ \kappa(t) &= \frac{|c'(t) \times c''(t)|}{|c'(t)|^3} \\ \kappa(t) &= \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} \\ \kappa'(t) &= \frac{-3ab(a^2-b^2)((a^2\sin^2 t + b^2\cos^2)^{\frac{1}{2}}}{(a^2\sin^2 t + b^2\cos^2 t)^3}\text{sin}t\text{cos}t \end{aligned}$$

Set the derivative equal to zero, we get $\text{sin}t = 0$ or cos t $= 0$. Plugging these points in $\kappa(t)$, we get the result.