Practice Problems

PROBLEM 1 Consider the surface $x^4 + (x^2 + y^2)z + z^3 = 1$.
(a) Determine for which points on the surface it is possible to solve for $z = g(x, y)$ locally.
(b) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(x, y, z) = (1, 2, 0)$.

PROBLEM 2 Suppose $c(t)$ is a path with velocity vector $v(t) = c'(t) = (3t\cos t, 5t\sin t, 4t\cos t)$ with $t \geq 0$.
(a) Calculate the unit tangent $T(t)$, normal $N(t)$, and curvature $\kappa(t)$, for $t > 0$.
(b) Is the path smooth at $t = 0$? Explain your answer.

PROBLEM 3 Let $F : \mathbb{R}^n \to \mathbb{R}$ be defined as $F(x) = ||x||^2$. Use the definition of differentiability to show that F is differentiable at every $x \in \mathbb{R}^n$, and $DF(x) = [2x_1 \; 2x_2 \; \cdots \; 2x_n]$.

PROBLEM 4 If $y = f(x)$ is twice differentiable, show its curvature is

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{\frac{3}{2}}}$$

Use the formula to find the curvature of $y = ln(\cos x)$. Show the curvature is zero in an inflection point.

PROBLEM 5 Find T, N, B, κ and τ, for the following curves:

$$\begin{aligned} r_1(t) &= (e^2\cos t, e^t\sin t, 2) \\ r_2(t) &= (\cosh t, -\sinh t, t) \\ r_3(t) &= (\cos t + t\sin t, \sin t - t\cos t, 3) \\ r_4(t) &= (\cos^3 t, \sin^3 t, 0) \end{aligned}$$

PROBLEM 6 Find the limit and prove it by definition:

$$1) \quad \lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x}$$

$$2) \quad \lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$$

$$3) \quad \lim_{(x,y)\to(0,0)} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$$

PROBLEM 7 Find the linearization at the given point:

$$\begin{aligned} f(x, y) &= e^x \cos y \quad (x, y) = (0, 0), (0, \frac{\pi}{2}) \\ g(x, y) &= (x + y + 2)^2 \quad (x, y) = (0, 0), (1, 2) \end{aligned}$$

PROBLEM 8 Find the points on the sphere $x^2 + y^2 + z^2 = 4$, that are closest to and farthest from the point $(3, 1, -1)$.

Problem 9 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

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\begin{aligned}
f_1(x,y,z) &= xyz, && x^2+2y^2+3z^2=6 \\
f_2(x,y,z,t) &= x+y+z+t, && x^2+y^2+z^2+t^2=1 \\
f_3(x,y,z) &= yz+xy, && xy=1,\ y^2+z^2=1
\end{aligned}
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