MATH 1K03 Final exam review

December 14, 2019

Question 1

Compute the limit.

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1}$$

Factor the numerator and denominator:

$$\lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)(x+1)}$$

Cancel the $(x-1)$ factor:

$$\lim_{x \to 1} \frac{(x+2)}{(x+1)}$$

Now use direct substitution $x = 1$:

$$\lim_{x \to 1} \frac{(x+2)}{(x+1)} = \frac{((1)+2)}{((1)+1)} = \frac{3}{2}$$

Question 2

Compute the limit. (Hint: rationalize the denominator)

$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$$

Multiply both the numerator and denominator by the conjugate of the denominator:

$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)$$

Distribute (FOIL) the denominator:

$$\lim_{x \to 4} \frac{(x-4)(\sqrt{x}+2)}{x-4}$$

Cancel the $(x-4)$ factor:

$$\lim_{x \to 4}(\sqrt{x}+2)$$

Now use direct substitution $x = 4$:

$$\lim_{x \to 4}(\sqrt{x}+2) = \sqrt{4}+2 = 2+2 = 4$$

Question 3

Choose the line through the point $(0, 1)$ which is **parallel** to

$$4x + 2y = 3$$

(A) x + 2y = 4 (B) 5x - y = 3 (C) 2x + 3y = 1 (D) 2x + y = 1

Put the given line in $y = mx + b$ form to identify its slope:

$$4x + 2y = 3$$

$$2y = -4x + 3$$

$$y = -2x + \frac{3}{2}$$

We see $m = -2$ is the slope of the given line. We want to find a new line parallel to the given line through the point $(0, 1)$. Parallel lines have the same slope, so the new line will also have $m = -2$.

Substituting $m = -2$, $x_0 = 0$, and $y_0 = 1$ into

$$y - y_0 = m(x - x_0)$$

gives

$$y - 1 = -2(x - 0)$$

or

$$2x + y = 1$$

so the answer is (D).
[Note: (D) is the only option with slope equal to -2. So you can deduce the answer without computing the full equation of the parallel line.]

Question 4

Choose the line through the point $(0, 1)$ which is **perpendicular** to

$$3x + 2y = 1$$

(A) 2y - 4x = 3 (B) 3y - 2x = 3 (C) 5y - x = -2 (D) 2x + y = 1

Put the given line in $y = mx + b$ form to identify its slope:

$$3x + 2y = 1$$

$$2y = -3x + 1$$

$$y = -\frac{3}{2}x + \frac{1}{2}$$

We see $m = -\frac{3}{2}$ is the slope of the given line. We want to find a new line perpendicular to the given line through the point $(0, 1)$. Perpendicular lines have slopes that are negative reciprocals of each other, so the new line will have slope $m = \frac{2}{3}$.

Substituting $m = \frac{2}{3}$, $x_0 = 0$, and $y_0 = 1$ into

$$y - y_0 = m(x - x_0)$$

gives

$$y - 1 = \frac{2}{3}(x - 0)$$

or

$$y - \frac{2}{3}x = 1$$

or (multiplying both sides by 3)

$$3y - 2x = 3$$

so the answer is (B).
[Note: (B) is the only option with slope equal to $\frac{2}{3}$. So you can deduce the answer without computing the full equation of the perpendicular line.]

Question 5

Find the constants A and B so that the graph of the function

$$f(x) = \frac{Ax + 7}{20 - Bx}$$

has $x = -2$ as a vertical asymptote and $y = 4$ as a horizontal asymptote.

Vertical asymptotes of $f(x)$ occur when the denominator $20 - Bx = 0$. If $x = -2$ is a vertical asymptote then

$$20 - B(-2)) = 0$$

$$20 + 2B = 0$$

$$B = -10$$

If $y = 4$ is a horizontal asymptote of $f(x)$ then it must be that

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{Ax + 7}{20 - Bx} = \frac{A}{-B} = 4$$

But we already know $B = -10$:

$$\frac{A}{-(-10)} = 4$$

$$\frac{A}{10} = 4$$

$$A = 40$$

So the constants $A = 40$ and $B = -10$ give us the desired asymptotes.

Question 6

Find the value of constant A, so that the following function is continuous for all x:

$$f(x) = \begin{cases} 5x - 3 & \text{if } x < 2 \\ Ax^2 + 4x + 3 & \text{if } x \geq 2 \end{cases}$$

The only potential point of discontinuity is $x = 2$. To make sure the function is continuous, we set both pieces of the function equal

$$5x - 3 = Ax^2 + 4x + 3$$

and then substitute $x = 2$:

$$5(2) - 3 = A(2^2) + 4(2) + 3$$

$$7 = 4A + 11$$

$$4A = -4$$

$$A = -1$$

Question 7

At a factory, the daily output is

$$Q(L) = 1200L^{2/3}$$

units, where L denotes the capital investment in dollars. The current capital investment is \$64,000. Estimate the additional capital ΔL required to increase output by 4500 daily units.

We use the approximation:

$$\Delta Q \approx Q'(L_0)\Delta L$$

which re-arranges to:

$$\Delta L \approx \frac{\Delta Q}{Q'(L_0)}$$

Compute the derivative:

$$Q'(L) = 1200\frac{2}{3}L^{-1/3} = 800L^{-1/3} = \frac{800}{L^{1/3}}$$

We are given: $L_0 = 64000$. So

$$Q'(L_0) = \frac{800}{(64000)^{1/3}} = \frac{800}{40} = 20$$

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We want to increase output by 4500, so $\Delta Q = 4500$. Thus

$$\Delta L \approx \frac{\Delta Q}{Q'(L_0)} = \frac{4500}{20} = 225$$

So to increase daily output by 4500 units, we require an increase of \$225 capital investment.

Question 8

Suppose that the total cost of producing x units of a particular commodity is given by

$$C(x) = 4x^2 - 5x + 324$$

Determine the level of production x at which the average cost $A(x) = C(x)/x$ is minimized.

First, write the average cost function: $A(x) = C(x)/x$:

$$A(x) = \frac{C(x)}{x} = \frac{4x^2 - 5x + 324}{x} = \frac{4x^2}{x} + \frac{-5x}{x} + \frac{324}{x}$$

Simplifying:

$$A(x) = 4x - 5 + 324x^{-1}$$

To minimize average cost, first compute its derivative:

$$A'(x) = 4 + (-1)324x^{-2} = 4 - \frac{324}{x^2}$$

$A(x)$ attains its minimum values when $A'(x) = 0$. This occurs when

$$0 = 4 - \frac{324}{x^2}$$

Solving for x we get

$$4 = \frac{324}{x^2}$$

$$4x^2 = 324$$

$$x^2 = \frac{324}{4} = 81$$

So $x = \pm\sqrt{81} = \pm 9$.

In this situation we only admit positive values for x. So average cost $A(x)$ is minimized when production level x is 9.

Question 9

Linear depreciation: Suppose \$1500 worth of books depreciate linearly in value to \$0 over 10 years. Express the value of the books as function of time. How much are the books worth after 5 years?

Let x correspond to time and y correspond to value. We are given two (x, y) points: $(0, 1500)$ and $(10, 0)$. Since we are told value in modelled linearly, our function will have the form of a line:

$$y = mx + b$$

Use the two points to compute the slope:

$$m = \frac{0 - 1500}{10 - 0} = \frac{-1500}{10} = -150$$

Since $(0, 1500)$ is a point on the line, we know the y-intercept $b = 1500$. So our function is:

$$y = -150x + 1500$$

Substitute $x = 5$ to compute the value after 5 years.

$$y(5) = -150(5) + 1500 = -750 + 1500 = 750$$

The books are worth \$750 after 5 years.

Question 10

Find the absolute maximum and absolute minimum of

$$f(x) = x^5 e^{-2x}$$

on the interval $[2, 5]$.

Recall: Extreme Value Property

Since $f(x)$ is continuous on closed interval $[a, b]$, the **Extreme Value Property** states that $f(x)$ attains its absolute maximum and minimum either at

- the boundary of the interval (a or b)
- a critical number c, (where $f'(c) = 0$), such that $a < c < b$.

Compute the derivative using the product rule:

$$f'(x) = 5x^4e^{-2x} + x^5(-2)e^{-2x}$$

To find the critical numbers, set $f'(x) = 0$ and solve for x:

$$0 = 5x^4e^{-2x} - 2x^5e^{-2x}$$

Factor:

$$0 = e^{-2x}x^4[5 - 2x]$$

Which gives $x = 0, \frac{5}{2}$ as critical numbers. [Note e^{-2x} is never 0.]
Since $x = 0$ is a critical number outside the interval $[2, 5]$ it cannot be an absolute min or max.

Now check the interval boundaries ($x = 2$ and $x = 5$) and critical numbers contained in the interval ($x = 2.5$):

$$f(2) = (2^5)e^{-4} \approx 0.5861$$
$$f(2.5) = (2.5^5)e^{-5} \approx 0.6580$$
$$f(5) = (5^5)e^{-10} \approx 0.1419$$

So $f(2.5) = (2.5^5)e^{-5}$ is the absolute max and $f(5) = (5^5)e^{-10}$ is the absolute min.

Question 11

Find an equation for the tangent line the to the graph of

$$f(x) = \ln(x^2)e^{2-x}$$

at the point where $x = 1$.

Compute the derivative using the product rule:

$$f'(x) = \frac{2x}{x^2}e^{2-x} + \ln(x^2)(-1)e^{2-x}$$

Simplifying:

$$f'(x) = \frac{2e^{2-x}}{x} - \ln(x^2)e^{2-x}$$

Find the slope when $x = 1$:

$$f'(1) = \frac{2e^{2-1}}{1} - \ln(1)e^{2-1} = 2e$$

[Recall $\ln(1) = 0$.]

We have the slope, now find a point:

$$f(1) = \ln(1)e^{2-1} = 0$$

So $(x_0, y_0) = (1, 0)$ is a point on the tangent line. Now sub this point and $m = 2e$ into

$$y - y_0 = m(x - x_0)$$

to get:

$$y - 0 = 2e(x - 1)$$

Simplfying the equation of the tangent line:

$$y = (2e)x - 2e$$

Question 12

Exponential model: Assume a population of kangaroos grows exponentially. If the population was 5000 in 1995 and it grew to 8000 in 2005, estimate the population in 2020.

The equation exponential growth is:

$$P(t) = P_0 e^{kt}$$

Let $t = 0$ correspond to 1995 with initial population $P_0 = 5000$. Let $t = 10$ correspond to 2005 with population $P(10) = 8000$.
Solve for the constant k:

$$P(10) = 8000 = 5000e^{10k}$$

which implies:

$$\frac{8}{5} = e^{10k}$$

Taking the ln of both sides, and cancelling ln and e:

$$\ln\left(\frac{8}{5}\right) = \ln\left(e^{10k}\right) = 10k$$

$$k = \frac{\ln\left(\frac{8}{5}\right)}{10} \approx 0.047$$

So we can estimate the population in 2020 by evaluating at $t = 25$:

$$P(25) = 5000e^{(0.047)25} \approx 16190$$

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Question 13

Logarithmic differentiation: Find $f'(x)$ when

$$f(x) = \sqrt[7]{\frac{5-2x}{2x+1}}$$

Recall:

$$\frac{d[\ln(f(x))]}{dx} = \frac{f'(x)}{f(x)}$$

Re-arranging:

$$f'(x) = f(x)\frac{d[\ln(f(x))]}{dx}$$

So to find $f'(x)$ we can compute $\frac{d[\ln(f(x))]}{dx}$ and multiply by $f(x)$.

Write $\ln(f(x))$:

$$\ln(f(x)) = \ln\left(\sqrt[7]{\frac{5-2x}{2x+1}}\right) = \ln\left(\left(\frac{5-2x}{2x+1}\right)^{\frac{1}{7}}\right)$$

Simplify using the power and quotient rules for *log*:

$$\ln\left(\left(\frac{5-2x}{2x+1}\right)^{\frac{1}{7}}\right) = \frac{1}{7}\ln\left(\frac{5-2x}{2x+1}\right) = \frac{1}{7}\Big[\ln(5-2x) - \ln(2x+1)\Big]$$

Taking the derivative:

$$\frac{d[\ln(f(x))]}{dx} = \frac{1}{7}\left[\frac{-2}{5-2x} - \frac{2}{2x+1}\right]$$

We now use

$$f'(x) = f(x)\frac{d[\ln(f(x))]}{dx}$$

to get

$$f'(x) = \sqrt[7]{\frac{5-2x}{2x+1}}\left(\frac{1}{7}\right)\left[\frac{-2}{5-2x} - \frac{2}{2x+1}\right]$$

Question 14

Find all real solutions x to the following equation

$$7^{3x} = \left(\frac{1}{49}\right)^{1-\frac{5}{2}x^2}$$

We want to re-write the right-hand side to have base 7:

$$7^{3x} = \left(\frac{1}{7^2}\right)^{1-\frac{5}{2}x^2}$$

$$7^{3x} = \left(7^{-2}\right)^{1-\frac{5}{2}x^2}$$

$$7^{3x} = 7^{(-2)(1-\frac{5}{2}x^2)}$$

Now the left-hand side and right-hand side have the same base. This implies their exponents are equal:

$$3x = (-2)(1 - \frac{5}{2}x^2)$$

We can solve for x now:

$$3x = -2 + 5x^2$$

$$5x^2 - 3x - 2 = 0$$

Using the quadratic formula (or otherwise) we get

$$x = 1, -\frac{2}{5}$$

as our solutions.

Question 15

Solve the following logarithmic equation for x:

$$\ln\left(\frac{12x^2}{x-2}\right) - 2\ln x = \ln 6$$

First re-write $2\ln(x)$ using the power rule:

$$\ln\left(\frac{12x^2}{x-2}\right) - \ln\left(x^2\right) = \ln 6$$

Now use the quotient rule on the left-hand side to get:

$$\ln\left(\frac{\left(\frac{12x^2}{x-2}\right)}{x^2}\right) = \ln 6$$

Cancelling the x^2, this simplifies to:

$$\ln\left(\frac{12}{x-2}\right) = \ln 6$$

which implies:

$$\frac{12}{x-2} = 6$$

and so

$$x = 4$$

Question 16

Use logarithmic rules to rewrite m in terms of $\log_3 2$ and $\log_3 5$:

$$m = \log_3 100$$

Since $100 = 10^2$ we can write:

$$m = \log_3(10^2)$$

The power rule for log gives:

$$m = 2\log_3(10)$$

Since $10 = 2 * 5$ we can write:

$$m = 2\log_3(2 * 5)$$

Then the product rule for log gives the result:

$$m = 2[\log_3(2) + \log_3(5)] = 2\log_3(2) + 2\log_3(5)$$

[You can check the result using $\log_b(a) = \frac{\ln(a)}{\ln(b)}$.]

Question 17

Effective interest rates: Which is better: an investment that earns 8.25% compounded quarterly, one that earns 8.20% compounded monthly, or one that earns 8.15% compounded daily?

The needed effective interest rate formula is

$$r_e = \left(1 + \frac{r}{k}\right)^k - 1$$

for interest rate r and yearly compounding frequency k.
For 8.25% and quaterly compounding, we get $r = 0.0825$ and $k = 4$ so

$$r_e = \left(1 + \frac{0.0825}{4}\right)^4 - 1 \approx 0.08509$$

For 8.20% and monthly compounding, we get $r = 0.082$ and $k = 12$ so

$$r_e = \left(1 + \frac{0.082}{12}\right)^{12} - 1 \approx 0.08515$$

For 8.15% and daily compounding, we get $r = 0.0815$ and $k = 365$ so

$$r_e = \left(1 + \frac{0.0815}{365}\right)^{365} - 1 \approx 0.08490$$

So 8.20% and monthly compounding is the best choice, since it has greatest r_e.

Question 18

Future value: Suppose $\$10,000$ is invested at an annual interest rate of 8% over 4 years. If x denotes the future value when interest is compounded semiannually and y is the future value when interest is compounded continuously, how much are x and y worth?

Future value (non-continuous):

$$B = P\left(1 + \frac{r}{k}\right)^{kt}$$

for initial value P, interest rate r, yearly compounding frequency k, and t in years. [Note "semiannual" means $k = 2$ here.]

$$x = 10000\left(1 + \frac{0.08}{2}\right)^{2*4} \approx \$13,685.69$$

Future value (continuous):

$$B = Pe^{rt}$$

for initial value P, interest rate r, and t in years.

$$y = 10000e^{(0.08*4)} \approx \$13,771.28$$