MATH 1K03 Midterm 3 review

November 26, 2019

Question 1

The first derivative of a certain function is

$$f'(x) = x(x-1)^2$$

a) On what intervals is f increasing? Decreasing?
b) Find the x-coordinate of any relative maximum and minimum of f.

(a) Increasing/decreasing depends on the sign of the first derivative:

- when $f'(x) > 0$, $f(x)$ is increasing
- when $f'(x) < 0$, $f(x)$ is decreasing

Compute the critical numbers (i.e. solve $f'(x) = 0$):

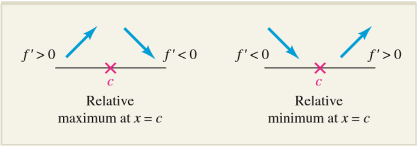
$f'(x) = 0$ exactly when $x = 0, 1$.

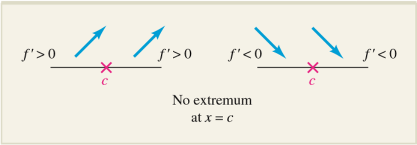
Divide the real number line into intervals determined by $x = 0, 1$ and test the sign of $f'(x)$:

x	(-∞, 0)	(0,1)	(1,+∞)
$f'(x)$	$-$	$+$	$+$
$f(x)$	decreasing	increasing	increasing

Since $f'(x)$ is negative on $(-\infty, -0)$, $f(x)$ is decreasing on $(-\infty, -0)$. Since $f'(x)$ is positive on $(0, 1)$ and $(1, +\infty)$, $f(x)$ is increasing on $(0, 1)$ and $(1, +\infty)$.

(b) To identify any relative maximum and minimum we use the first derivative test for relative extreme. If $f'(c) = 0$ of $f'(c)$ does not exist then





In other words, the sign of $f'(x)$ must change around the critical number.

(b)

x	(-∞, 0)	(0,1)	(1,+∞)
$f'(x)$	$-$	$+$	$+$
$f(x)$	decreasing	increasing	increasing

In this case the sign of $f'(x)$ changes from negative to positive around the critical number $x = 0$. Since the sign of $f'(x)$ does not change around $x = 1$ is is not an extremum.

We conclude $x = 0$ is the x-coordinate of a **relative minimum** of $f(x)$.

Question 2

Let $f(x) = x^3 - 3x^2 + 7$. Find the absolute maximum and minimum of $f(x)$ on the interval $[-1, 3]$. (i.e. for $-1 \leq x \leq 3$).

Recall: Extreme Value Property

Since $f(x)$ is continuous on closed interval $[a, b]$, the **Extreme Value Property** states that $f(x)$ attains its absolute maximum and minimum either at

- the boundary of the interval (a or b)
- a critical number c, (where $f'(c) = 0$), such that $a < c < b$.

To find the critical numbers of $f(x)$, compute its derivative:

$$f'(x) = 3x^2 - 6x$$

And factor the derivative:

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

to see that $f'(x) = 0$ when $x = 0, 2$ (critical numbers of $f(x)$).

Note that both critical numbers are contained in the interval $[-1, 3]$. The boundary points, along with the critical numbers in the interval, give a list of x-coordinate candidates to induce an absolute min/max:

$$x = -1, 0, 2, 3$$

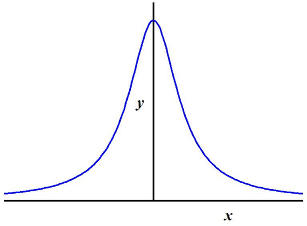
Evaluating each candidate: f(-1)=3 f(0)=7 f(2)=3 f(3)=7

Taking the greatest and least values we conclude:

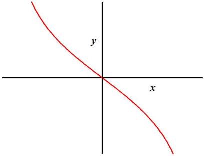
- 7 is the **absolute maximum** of $f(x)$ on $[-1, 3]$
- 3 is the **absolute minimum** of $f(x)$ on $[-1, 3]$.

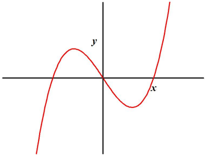
Question 3

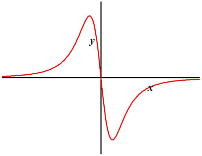
The graph of $f(x)$ is given in blue.

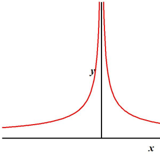


Which of the below graphs is a graph of its derivative $f(x)'$?

(A) 

(B) 

(C) 

(D) 

Notice $f(x)$ is flat (has horizontal tangent) at $x = 0$. This means the derivative $f'(x)$ must be zero at $x = 0$. (A), (B), and (C) satisfy this condition. Since (D) fails this condition the answer cannot be (D).

Notice $f(x)$ is increasing for $x < 0$ and decreasing for $x > 0$. This means the derivative $f'(x)$ must be positive for $x < 0$ and negative for $x > 0$. (A) and (C) satisfy this condition but (B) fails it. So the answer cannot be (B).

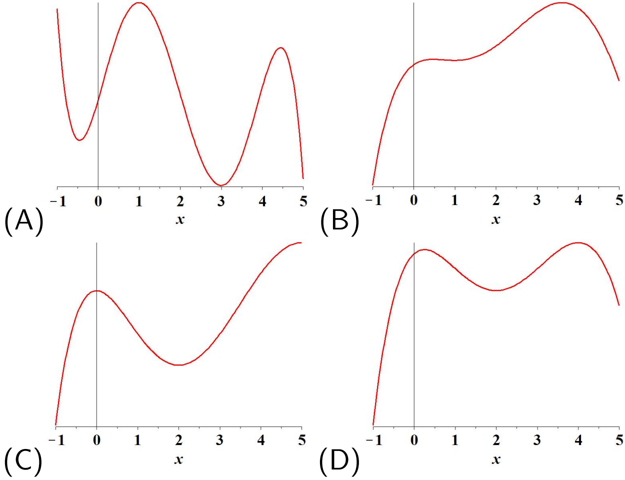
Notice as $x \to +\infty$ and $x \to -\infty$ the tangent lines to $f(x)$ approach being horizontal. This means the derivative $f'(x)$ should approach zero as x gets large. (C) satisfies this condition but (A) fails it. So the answer cannot be (A).

We conclude the answer is (C).

Question 4

Which of the below graphs is an example of a function that satisfies the following conditions?

$g'(0) = 1, g'(1) = 0, g'(2) = 1, g'(3) = 1, g'(4) = -1$



The condition $g'(0) = 1$ is satisfied by (A), (B), and (D). The condition is not satisfied by (C) since the graph is flat at $x = 0$ (so the derivative should be 0). So the answer cannot be (C).

The condition $g'(1) = 0$ is satisfied by (A) and (B). The condition is not satisfied by (D) since the graph is decreasing at $x = 1$ (so the derivative should be negative). So the answer cannot be (D).

The condition $g'(2) = 1$ is satisfied by (B). The condition is not satisfied by (A) since the graph is decreasing at $x = 2$ (so the derivative should be negative). So the answer cannot be (A).

We conclude the answer is (B).

Question 5

Determine all values of x where the following function is concave up.

$$f(x) = \frac{1}{12}x^4 + x^3 - 8x^2 - 5x + 17$$

Concavity depends on the sign of the second derivative:

- when $f''(x) > 0$, $f(x)$ is concave up
- when $f''(x) < 0$, $f(x)$ is concave down

Compute the first derivative:

$$f'(x) = \frac{1}{3}x^3 + 3x^2 - 16x - 5$$

Compute the second derivative:

$$f''(x) = x^2 + 6x - 16$$

Factor the second derivative:

$$f''(x) = x^2 + 6x - 16 = (x + 8)(x - 2)$$

So $f''(x) = 0$ exactly when $x = -8, 2$.

Divide the real number line into intervals determined by $x = -8, 2$ and test the sign of $f''(x)$:

x	(-∞, -8)	(-8,2)	(2,+∞)
$f''(x)$	$+$	$-$	$+$
$f(x)$	concave UP	concave DOWN	concave UP

Since $f''(x)$ is positive on $(-\infty, -8)$ and $(2, +\infty)$, we conclude $f(x)$ is concave up on $(-\infty, -8)$ and $(2, +\infty)$.

Question 6

Suppose that the second derivative of a function $f(x)$ is given by

$$f''(x) = x^2(x-2)^7(x-3)^2$$

Find the x-coordinates of the inflection points of $f(x)$ (if any).

An inflection point is a point on the graph of a function f where f is continuous and the concavity changes.

So we looking x-coordinates around which $f''(x)$ changes sign.

$f''(x) = 0$ exactly when $x = 0, 2, 3$.

Divide the real number line into intervals determined by $x = 0, 2, 3$ and test the sign of $f''(x)$:

x	$(-\infty, 0)$	$(0,2)$	$(2,3)$	$(3,+\infty)$
$f''(x)$	$-$	$-$	$+$	$+$
$f(x)$	concave DOWN	concave DOWN	concave UP	concave UP

- Around $x = 0$, $f''(x)$ does not change sign.
- Around $x = 3$, $f''(x)$ does not change sign.
- Around $x = 2$, $f''(x)$ changes negative to positive. So $x = 2$ is the x-coordinate of an inflection point of $f(x)$.

Question 7

Suppose that the price at which x units of a particular commodity can be sold is given by

$$p(x) = 382 - 3x$$

and that the total cost of producing x units is

$$C(x) = 7x^2 + 4x + 703$$

(a) Find the level of production x at which the profit function $P(x)$ is maximized.
(b) Determine the level of production x at which the average cost $A(x) = C(x)/x$ is minimized.

(a) To compute the profit function, we first need the revenue function. In general the revenue is the number of items sold multiplied by the selling price. In this case:

$$R(x) = xp(x)$$

$$R(x) = x(382 - 3x) = 382x - 3x^2$$

The profit function is then:

$$P(x) = R(x) - C(x)$$

$$P(x) = 382x - 3x^2 - [7x^2 + 4x + 703]$$

$$P(x) = 382x - 3x^2 - 7x^2 - 4x - 703$$

which simplifies to

$$P(x) = -10x^2 + 378x - 703$$

To maximize profit, first compute its derivative:

$$P'(x) = -20x + 378$$

$P(x)$ attains its maximum values when $P'(x) = 0$. This occurs when

$$0 = -20x + 378$$

or when $x = \frac{378}{20} \approx 18.90$.

So profit $P(x)$ is maximized when production level x is approximately 18.90.

(b) First, write the average cost function: $A(x) = C(x)/x$:

$$A(x) = \frac{C(x)}{x} = \frac{7x^2 + 4x + 703}{x} = \frac{7x^2}{x} + \frac{4x}{x} + \frac{703}{x}$$

Simplifying:

$$A(x) = 7x + 4 + 703x^{-1}$$

To minimize average cost, first compute its derivative:

$$A'(x) = 7 + (-1)703x^{-2} = 7 - \frac{703}{x^2}$$

$A(x)$ attains its minimum values when $A'(x) = 0$. This occurs when

$$0 = 7 - \frac{703}{x^2}$$

Solving for x we get

$$7 = \frac{703}{x^2}$$

$$7x^2 = 703$$

$$x^2 = \frac{703}{7}$$

So $x = \pm\sqrt{\frac{703}{7}} \approx \pm 10.02$.

In this situation we only admit positive values for x. So average cost $A(x)$ is minimized when production level x is approximately 10.02.

Question 8

Find the constants A and B so that the graph of the function

$$f(x) = \frac{Ax + 5}{6 - Bx}$$

has $x = 3$ as a vertical asymptote and $y = -4$ as a horizontal asymptote.

Vertical asymptotes of $f(x)$ occur when the denominator $6 - Bx = 0$. If $x = 3$ is a vertical asymptote then

$$6 - B(3) = 0$$

$$B = 2$$

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If $y = -4$ is a horizontal asymptote of $f(x)$ then it must be that

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{Ax + 5}{6 - Bx} = \frac{A}{-B} = -4$$

But we already know $B = 2$:

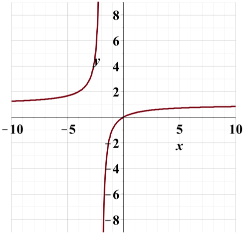
$$\frac{A}{-2} = -4$$
$$A = 8$$

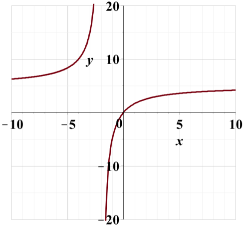
So the constants $A = 8$ and $B = -2$ give us the desired asymptotes.

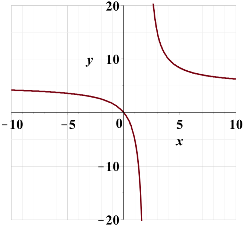
Question 9

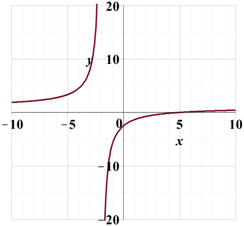
Identify the graph that best matches the function

$$f(x) = \frac{5x}{x+2}$$

(A) 

(B) 

(C) 

(D) 

First notice $f(x)$ has a vertical asymptote when $x = -2$. This means the answer cannot be (C).

Compute the horizontal asymptote of $f(x)$:

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{5x}{x+2} = \frac{5}{1} = 5$$

So $f(x)$ has horizontal asymptote $y = 5$. This means the answer cannot be (A) or (D).

We conclude graph (B) best matches $f(x)$.

Question 10

Find two positive numbers x and y whose sum is 50 and whose product is a large as possible.

Relation: x and y must satisfy $x + y = 50$.
Maximize: xy
xy is a function of two variables x and y. We want to use the relation $x + y = 50$ to re-write xy as a function of one variable.

$$x + y = 50$$

can be re-written as

$$y = 50 - x$$

Now substitute $y = 50 - x$ into xy to get

$$f(x) = x(50 - x)$$

which is a function on one variable x.

Maximize: $f(x) = x(50 - x) = 50x - x^2$
Compute the derivative:

$$f'(x) = 50 - 2x$$

$f(x)$ is maximized when $f'(x) = 0$. So solve

$$0 = 50 - 2x$$

which gives $x = 25$.
When $x = 25$, we can use the relation $y = 50 - x$ to find the corresponding y-value.

$$y = 50 - 25 = 25$$

The (x, y) pair $(25, 25)$ are the positive numbers that maximize their product xy while satisfying $x + y = 50$.

Question 11

A closed box with a square base is to have a volume of $192m^3$. The material for the top and bottom of the box costs $3/m^2$, and the material for the sides costs $1/m^2$. Find the minimum possible cost for a such a box.

Suppose the box has dimensions x, y, h. Since the base is square, set $x = y$. Then the area of each side piece is xh and the area of the top/bottom is x^2.
Cost of one side: $1xh$
Cost of top/bottom: $3x^2$
Total cost of box: $4(1xh) + 2(3x^2) = 4xh + 6x^2$
We want to minimize the cost of the box, so we need to re-write it as a function of one variable.

Volume of box $= hx^2 = 192$ which implies

$$h = \frac{192}{x^2}$$

Now substitute $h = \frac{192}{x^2}$ into $4xh + 6x^2$ to get

$$C(x) = 4x\frac{192}{x^2} + 6x^2$$

which simplifies to

$$C(x) = \frac{768}{x} + 6x^2 = 768x^{-1} + 6x^2$$

Since cost is now a function a single variable x, we can compute its derivative:

$$C'(x) = (-1)768x^{-2} + 12x = -\frac{768}{x^2} + 12x$$

To minimize the cost, we solve $C'(x) = 0$:

$$0 = -\frac{768}{x^2} + 12x$$

$$12x = \frac{768}{x^2}$$

$$12x^3 = 768$$

$$x^3 = \frac{768}{12} = 64$$

$$x = \sqrt[3]{64} = 4$$

So the cost $C(x)$ is minimized when $x = 4$. We find the minimal cost of the box by computing

$$C(4) = \frac{768}{4} + 6(4)^2 = 192 + 96 = 288$$

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