MATH 3MB3 FALL 2018 LINEAR UNIVARIATE CONTINUOUS DETERMINISTIC

The basic model is now
$\frac{dx(t)}{dt} = rx(t)$
where r is a constant and $\frac{d}{dt}$ is the differential operator associated with a derivative with respect to time. Rather than the recursion equation we had for discrete-time models, the equation at hand is a first order ordinary differential equation (ODE). It can be thought of as the limit of the discrete-time linear model as the time step gets smaller, i.e. from the discrete model $x(t+h) = (1+rh)x(t)$

$$\frac{dx}{dt} = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}$$
$$= \lim_{h\to 0} \frac{(1+rh)x(t) - x(t)}{h}$$
$$= \lim_{h\to 0} \frac{(rh)x(t)}{h}$$
$$= \lim_{h\to 0} \frac{(rh)}{h}x(t)$$
$$= \lim_{h\to 0} rx(t)$$
$$= rx(t)$$

1. Fixed points and stability

A point is still a fixed point only if when the model is applied to that fixed point, the fixed point doesn't change. In the case of this continuous model, $\frac{dx}{dt}$ can be thought of as the change in x with respect to time. Hence x^* will be a fixed point of the model only if $\frac{dx}{dt}$ evaluated at $x^* = 0$. In the above example we require that $rx^* =$ which means that if $r \neq 0$ then $x^* = 0$. Just like before with the linear case, if $r = 0$ then any/every real number is a fixed point.

2. stability

Stability is handled different compared to the linear model. For this example if we want x^* to be stable, then we require both that x in increase for $x < 0$, and x is decreasing for $x > 0$ (x^* has to be attracting from both sides).
If we look at $x > 0$ we get that $\frac{dx}{dt} = rx$, and if we want this to be decreasing we require that $r < 0$, since $x > 0$.

If we look at $x > 0$ we get that $\frac{dx}{dt} = rx$, and if we want this to be increasing we require that $r < 0$, since $x < 0$.
Therefore for this example, $x^* = 0$ is stable if $r < 0$.

3. Explicit solution in time

The method for solving a first order ODE is using the method of separation of variables. This involves taking all independent variables to one side of the equation, while taking the dependent variables to the other side. Writing $x = x(t)$ we obtain the following:
$\frac{dx}{dt} = rx$
$\frac{dx}{x} = rdt$
$\int_{x(0)}^{x(t)} \frac{1}{x}dx = r\int_0^t 1dt$
$ln(x(t)) - ln(x(0)) = rt$ (assuming $x(0) > 0$)
$x(t) = x(0)e^{rt}$
The last formula provides us with the general formula for $x(t)$ given an initial condition $x(0)$.

4. Affine Models

A continuous (more realistic) leaky bucket model is $\frac{dx}{dt} = a - bx$. The fixed points are found from $0 = a - bx^*$ which is $\frac{a}{b}$ assuming that $b \neq 0$. This equilibrium is stable whenever $b > 0$ (note that b has the opposite sign of r in the linear model). The explicit solution of this ODE can be found by separation of variables to be
$x(t) = e^{-bt}(x(0) - \frac{a}{b}) + \frac{a}{b}$