Math 3MB3 midterm test, 16 October 2017

Name:

Reminders

- Don't forget to `import numpy as np` if you need it in your Python code.
- $$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & 0 \\ -b/(ac) & 1/c \end{pmatrix}$$
- Absorbing state transient occupancy: $\mathbf{F} = (\mathbf{1} - \mathbf{A})^{-1}$. Absorbing state probabilities: $\mathbf{BF}$.
- Jury conditions: a 2-D MNDD system is stable if $|T| < 1 + \Delta < 2$.

Python (24 points/12 points each)

Hints: you will need to use `for` loops in these problems since they don't have closed-form solutions. Use `numpy.zeros(n)` to initialize a `numpy` array of length `n`.

1. Write Python code to compute the numerical solution of the Ricker equation $x(t+1) = ax\exp(-bx(t))$ for $x(9)$, starting from $x(0) = 0.1$, with $a = 2$ and $b = 1$.

```
x=0.1
a, b = 2, 1 # unpacking (unnecessarily fancy)
for t in range(10):
    x = a*x*np.exp(-b*x)
```

2. Write Python code to compute the numerical solution of the Nicholson-Bailey equations:

$$V_{t+1} = rV_t e^{-qP_t}$$
$$P_{t+1} = cV_t(1 - \exp(-qP_t)$$

starting from $\{V = 1, P = 1\}$, with parameters $r = 2$, $q = 1$, $c = 1$, for 100 steps. Save the results for all time steps in two `numpy` arrays `P` and `V`.

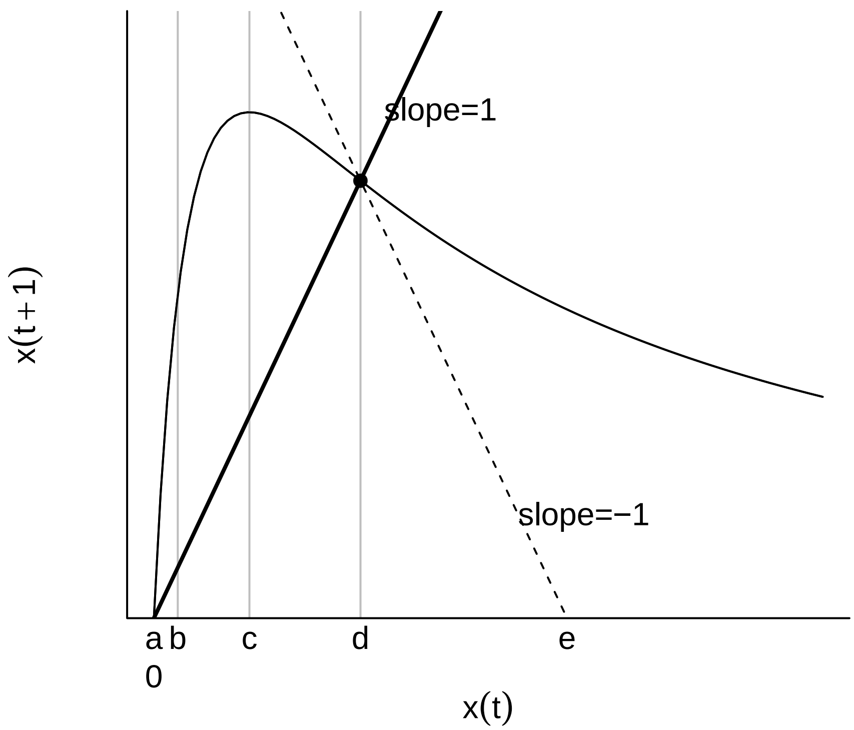
```
import numpy as np
P = np.zeros(100)
V = np.zeros(100)
r, q, c = 2, 1, 1 # unpacking (unnecessarily fancy)
for t in range(99):
    V[t+1] = r*V[t]*np.exp(-q*P[t])
    P[t+1] = c*V[t]*(1-np.exp(-q*P[t]))
```

Equilibria and stability of 1-D systems (24 points/6 points each)

Consider the following UNDD system:

$$X_{t+1} = RX_t/(1+X_t)^2$$

(assume $R > 0$, $X \geq 0$).
3. Find the equilibrium or equilibria (if more than one) analytically in terms of R.
4. Find the stability criterion for the *simplest* equilibrium (you choose which one).
Using the diagram below:
5. identify which of the points a-e are equilibria of the system and state their stability. Explain your reasoning.
6. Draw the cobweb diagram representing the dynamics starting from $x(0) = b$.



Equilibria:

$$
\begin{aligned}
x^* &= Rx^*/(1+(x^*)^2) \\
x^* &= 0 \quad \textbf{or} \\
1 &= R/(1+(x^*)^2) \\
(1+(x^*)^2) &= R \\
(x^*)^2 &= R-1 \\
x^* &= \pm\sqrt{R-1}
\end{aligned}
$$

(it's OK to write down just the positive square root)

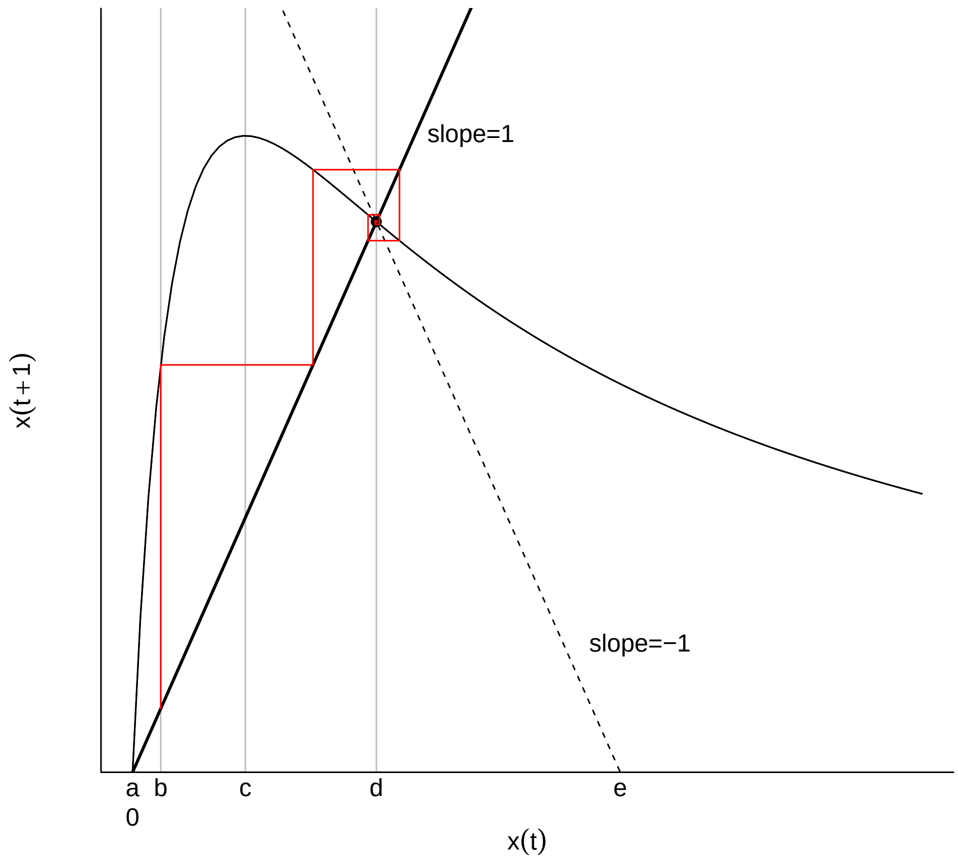
Stability:

$$
\begin{aligned}
f'(x) &= R\left(\frac{(1+(x^*)^2) - x^* \cdot 2x^*}{(1+(x^*)^2)^2}\right) \\
&= R\left(\frac{(1-(x^*)^2)}{(1+(x^*)^2)^2}\right)
\end{aligned}
$$

At $x^* = 0$: $f'(x) = R$, so stable if $R < 1$ (because we assumed $R > 0$).
(**Not required**) At $x^* = \sqrt{R-1}$:

$$
\begin{aligned}
f'(x) &= (1-(R-1))/(1+(R-1))^2 \\
&= (2-R)/R^2
\end{aligned}
$$

The derivative of this criterion is $f''(x) = -4/R^3 + 2/R^2 = 2/R^2(-2/R+1)$ So the value is decreasing from $R = 0$ to $R = 2$, increasing thereafter. The value is 1 at $R = 1$. The minimum value is -1/2. So $|f'(x)| < 1$ is true as long as $R > 1$. So this equilibrium is stable whenever the zero equilibrium is unstable.



Equilibria and stability of 2D systems (24 points/12 points each)

For the two-dimensional epidemic model,

$$S_{t+1} = S_t + mN - mS_t - \beta S_t I_t$$
$$I_{t+1} = I_t + \beta S_t I_t - (m + \gamma) I_t$$

You can assume m, N, β, γ are positive.
7. Find the two equilibria. (*Hint*: start with the I equation.)

8. Compute the Jacobian. Evaluate it at the disease-free equilibrium ($I^* = 0$) and show that if $\beta N = \gamma$ and $0 < m < 1$, the equilibrium is **neutrally stable**, i.e. that $T = 1 + \Delta$ and $\Delta < 1$ (where T=trace, Δ=determinant of the Jacobian).

Equilibria: (1) $I^* = 0$ is a solution of the second equation; in this case $S^* = S^* + mN - mS^*$ so $S^* = N$.
(2) Divide I equation by I^* to get $1 = 1+\beta S^* - (m+\gamma)$ so $S^* = (m+\gamma)/beta$. Divide S^* equation by S^* and substitute:

$$\begin{aligned} 1 &= (1-m) + mN/S^* - \beta I^* \\ \beta I^* &= mN/S^* - m \\ I^* &= m(N/S^* - 1)/\beta = mN/(m+\gamma) - 1/\beta \end{aligned}$$

Jacobian:

$$\begin{pmatrix} 1 - m - \beta I^* & -\beta S^* \\ \beta I^* & 1 - (m+\gamma) + \beta S^* \end{pmatrix}$$

At $S^* = N$, $I^* = 0$:

$$\begin{pmatrix} 1 - m & -\beta N \\ 0 & 1 - (m+\gamma) + \beta N \end{pmatrix}$$

Trace: $2 - 2m - \gamma + \beta N$; Det: $(1-m)(1-(m+\gamma)+\beta N)$
If $\beta N = \gamma$ we have $T = 2 - 2m$, $\Delta = (1-m)(1-m)$.
Stable if

$$|2 - 2m - \gamma + \beta N| < 1 + (1-m)(1-(m+\gamma)+\beta N) < 2$$

It turns out that the determinant condition is always true if the trace condition is, and the trace condition is violated if $\beta N/\gamma > 1$.

Markov models (28 points/7 points each)

At a party, bottles of beer have four possible states: (a) unopened, (b) open and in the process of consumption, (c) empty, and (d) "dead", i.e. bottles that are opened but have been abandoned by their owner while some beer remains (no-one will pick them up and drink them after this).

The transition matrix for the system is as follows:

$$\mathbf{M} = \begin{array}{l} \text{unopened} \\ \text{open} \\ \text{empty} \\ \text{dead} \end{array} \overset{\begin{array}{cccc} \text{unopened} & \text{open} & \text{empty} & \text{dead} \end{array}}{\begin{pmatrix} 0.2 & 0 & 0 & 0 \\ 0.8 & 0.5 & 0 & 0 \\ 0 & 0.3 & 1 & 0 \\ 0 & 0.2 & 0 & 1 \end{pmatrix}}$$

9. Draw a state diagram for the system. Include labels for the states; arrows indicating the direction of flows; and labels on the arrows indicating the *per capita* flow rate.
10. What are the absorbing states? How do you know?
11. What is the probability of ending up in each of the absorbing states, depending on which of the non-absorbing states a bottle starts in?
12. What is the total expected number of time steps before ending up undrinkable (i.e., in any one of the absorbing states) based on starting out (a) unopened or (b) open?

Absorbing states: empty and dead (columns containing all zero values except for the diagonal element, which is equal to 1)

$$\mathbf{A} = \begin{pmatrix} 0.2 & 0 \\ 0.8 & 0.5 \end{pmatrix}$$

$$\begin{aligned} \mathbf{F} &= (\mathbf{1} - \mathbf{A})^{-1} \\ &= \begin{pmatrix} 0.8 & 0 \\ -0.8 & 0.5 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{0.8} & 0 \\ \frac{0.8}{0.8 \cdot 0.5} & \frac{1}{0.5} \end{pmatrix} \\ &= \begin{pmatrix} 1.25 & 0 \\ 2 & 2 \end{pmatrix} \end{aligned}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0.3 \\ 0 & 0.2 \end{pmatrix}$$

$$\mathbf{BF} = \begin{pmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{pmatrix}$$

From **BF**: $P(\text{empty}|\text{unopened}) = P(\text{empty}|\text{open}) = 0.6$; $P(\text{dead}|\text{unopened}) = P(\text{dead}|\text{open}) = 0.4$.
From $\sum_i \mathbf{F}_{ij}$: total number of time steps from unopened = 3.25. Total number of time steps from open = 2.

The End