MATH 3MB3 FALL 2018 NONLINEAR UNIVARIATE DISCRETE DETERMINISTIC

A general nonlinear first order recursion follows the same formula as before with $N(t+1) = f(N(t))$, however in this case f is a nonlinear function. The nonlinearity of f makes for a more difficult analysis in most cases, but it also makes this model much more versatile than the linear one.

1. Logistic Model

The logistic model involves imposing bounds on the unrealistic exponential growth process that occurs in linear models. Begin with the geometric difference equation, $N(t+1) - N(t) = rN(t)$. Now set r equal to a decreasing linear function of $N(t)$ with x-intercept K and y-intercept R. This yields the difference equation
$N(t+1) - N(t) = \frac{R}{K}(K - N(t))N(t)$
which in turn leads to the logistic equation
$N(t+1) - N(t)(R(1 - \frac{N(t)}{K}) + 1)$

Fixed Points. Just like with the linear models we have done so far we start by sticking in $N^* = N(t) = N(t+1)$. This gets us the equation:
$N^* = N^*(R(1 - \frac{N^*}{K}) + 1)$
This yields two fixed points. $N^* = 0$, which corresponds to the case where there is zero population, and $N^* = K$, which corresponds to a sustainable non-zero population, which is noted as the carrying capacity.

Stability. As mentioned in the previous lecture, a criterion for stability is that $|f'(N^*)| < 1$ where $f'(N)$ is the first derivative of f with respect to N. For the logistic recursion discussed here, we have $f(N) = N(R(1 - \frac{N}{K}) + 1)$. The derivative of this function is
$f'(N) = R + 1 - \frac{2RN}{K}$

The first equilibrium $N^* = 0$ has $f'(0) = R + 1$ so it will be stable if $|R + 1| < 1$ which is equivalent to $-2 < R < 0$. The second has $f'(K) = 1 - R$ so it will be stable if $|1 - R| < 1$ which is equivalent to $0 < R < 2$. One thing to notice is that neither of these stability ranges overlap, so they can not both be stable for a given value of R

If, for example, one is modelling a population of creatures for which it is known that $R = 1$, one can conclude that any non-zero population size will tend towards the carrying capacity. We found the long term dynamics for this system without even having to solve for N(t).

2. Alternative parameterizations

An ecologist might choose to parameterize the discrete logistic model as above. A mathematician would probably write
$x(t+1) = Ax(t)(B - x(t))$.
The mathematician has chosen $A = \frac{R}{K}$ and $B = K + \frac{K}{R}$. Mathematically equivalent parameterizations often have quite different meanings (or statistical properties), as well as cultural connotations. Another way of thinking about this is if you had a constant birth rate b and a constant death rate d, you could either model it as $N(t+1) = bN(t) - dN(t)$ or $N(t+1) = wN(t)$ where w is $b - d$.

3. More nonlinear models

These discrete models often come up as approximations of continuous ones. Other 1-D discrete nonlinear models exist, such as the Ricker model
$N(t+1) = rN(t)e^{bN(t)}$
which was first introduced as a model for fisheries, and also epidemic models, such as the SI model, which is similar to discrete logistics.

We examine the SI model more closely. Let S(t) represent the number of people susceptible to a disease, I(t) the number of infected and N the (time-independent) total population. We now make several assumptions. First, that everyone is either susceptible or infected ($N = S(t) + I(t)$). Second, that susceptible individuals catch the disease from infected individuals through a contact rate β. Third, that infected individuals recover at a rate of γ. Then
$S(t+1) - S(t) = -\beta S(t)I(t) + \gamma I(t)$.
Note that the model right now is really bivariate. To make it univariate, recall that $I(t) = N - S(t)$ so that:
$S(t+1) = S(t) + (\gamma - \beta S(t))(N - S(t))$.
This is a slight modification of the logistic equation! You have all the tools to analyse it yourself.

More involved epidemic models exist. Two examples are the SIR and SEIR model, which we will study later.