MATH 3MB3 FALL 2018 HOMEWORK 2
Due Monday October 15 at 11:59PM

Format your report using some form of word processing software (Word, Latex, OpenOffice, ...), export it to a PDF file and submit it via email to:

- Alexandra Bushby, bushbya@mcmaster.ca if your last name starts with A-G or if you are submitting using R
- Robert White, whitere@mcmaster.ca if your last name starts with a H-Z, or you plan on using Python, and you don't plan on using R

together with a file containing the code you used for your computer simulations.
Make sure that your email has the proper subject line and information from the outline.

Question 1

Lets let A be the number of young adults in a population and E be the number of elderly people in the population. We assume that the birth rate for young adults is b, and the birth rate for elderly is 0. Furthermore we assume that survival rates for adults and elderly are s_A, s_E respectively. It is further assumed that if a young adult survives for the year they magically become elderly. Recall that all rates are always between 0 and 1.
a) Find the eigenvalues

b) Find the eigenvectors (do not simplify the complicated one)

c) Use a) and b) to setup the explicit solution given initial values $A(0) = A_0$ and $E(0) = E_0$. Do NOT solve for C_1, C_2

For the rest of question 1) use computer software and set $s_A = \frac{1}{10}$, $s_E = \frac{1}{20}$, $b = \frac{1}{4}$, $A(0) = 50$, $E(0) = 20$:
d) What is the explicit solution now?

e) Calculate $A(10)$ and $E(10)$

f) Describe what you expect to happen to A and E over time. Justify using b), c), d), e).

g) Plot A and E as functions of time (at least till time 10), and describe their long term dynamics

19 marks for question 1
2 marks for a)
4 marks for b)
4 marks for c)
3 marks for d)
2 marks for e)
2 marks for f)
2 marks for g)

a) $\begin{bmatrix} A(t+1) \\ E(t+1) \end{bmatrix} = \begin{bmatrix} b & 0 \\ s_A & s_E \end{bmatrix} \begin{bmatrix} A(t) \\ E(t) \end{bmatrix}$

Eigenvalues can be read off the diagonal. b and s_E
1 mark for setting up the matrix
1 mark for correct eigenvalues

b)Let $\lambda_1 = b$, then (A-bI) $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$=0 becomes

$\begin{bmatrix} 0 & 0 \\ s_A & s_E - b \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

which becomes $0 = 0$ and $s_A v_1 + (s_e - b)v_2 = v_2$
This is the non trivial eigenvalue and they can simply just state the above formula for full marks for this part

$(A - s_E I) \begin{bmatrix} v_3 \\ v_4 \end{bmatrix}$=0 becomes

$\begin{bmatrix} b - s_E & 0 \\ s_A & 0 \end{bmatrix} \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ v_4 \end{bmatrix}$

which becomes $(b - s_E)v_3 = 0$ and $s_A v_3 = 0$
Therefore $v_3 = 0$ and $v_4 = 1$
1 mark for the setup of $A - bI$
1 mark for setting up both with the correct eigenvalues
1 mark for getting the correct 2 equations of each
1 mark for solving second one as (0,1).

c) $x(t) = C_1(b)^t(v_1, v_2) + C_2(s_E)^t(0, 1)$
1 mark for still having the C_i's
1 mark for replacing the λ's by the correct eigenvalue
1 mark for the (0,1) in the 2nd part
1 mark for the rest being correct

d) Giant block of example code:

```
s_A=1/10;\\
s_E=1/20;\\
b=1/4;\\
A=[b 0; s_A, s_E];\\
[C,D]=eig(A);\\
A0=50;\\
E0=20;\\
t=0;% used in solving C1,C2\\
C1=A0*5/2/5^(1/2);\\
C2=E0-5^(1/2)*C1/5;\\
C1*b^t*C(1,2)+C2*s_E^t*C(1,1);%=A0;\\
%(2*5^(1/2)*C1)/5=A0\\
%C1=A0*5/2/5^(1/2)~=55.9017\\
C1*b^t*C(2,2)+C2*s_E^t*C(2,1);%=E0;\\
%(C2 + (5^(1/2)*C1)/5=E0\\
%C2=E0-5^(1/2)C1/5\\
%C2=-5\\
b=sym('b');\\
sE=sym('sE');\\
t=sym('t');\\
C1*b^t*C(1,2)+C2*sE^t*C(1,1)\\
C1*b^t*C(2,2)+C2*sE^t*C(2,1)\\
```

$A(t) = 50(\frac{1}{4})^t$
$E(t) = 25\frac{1}{4}^t - 5\frac{1}{20}^t$
This one will be hard to mark.
If they get the answer correct, full 4 marks. If they get the answer wrong you may have to look at their code to even see what they did.
1 mark for Using [C,D]=eig(A) to get the eigenvectors of A
1 mark for solving for C1,C2
1 mark for then using the solved C1,C2 to get the answer

e)

```
C1*b^t*C(1,2)+C2*s_E^t*C(1,1)
C1*b^t*C(2,2)+C2*s_E^t*C(2,1)
```

4.7684e-05
2.3842e-05

4.768371582031250e-05
2.384185742187500e-05

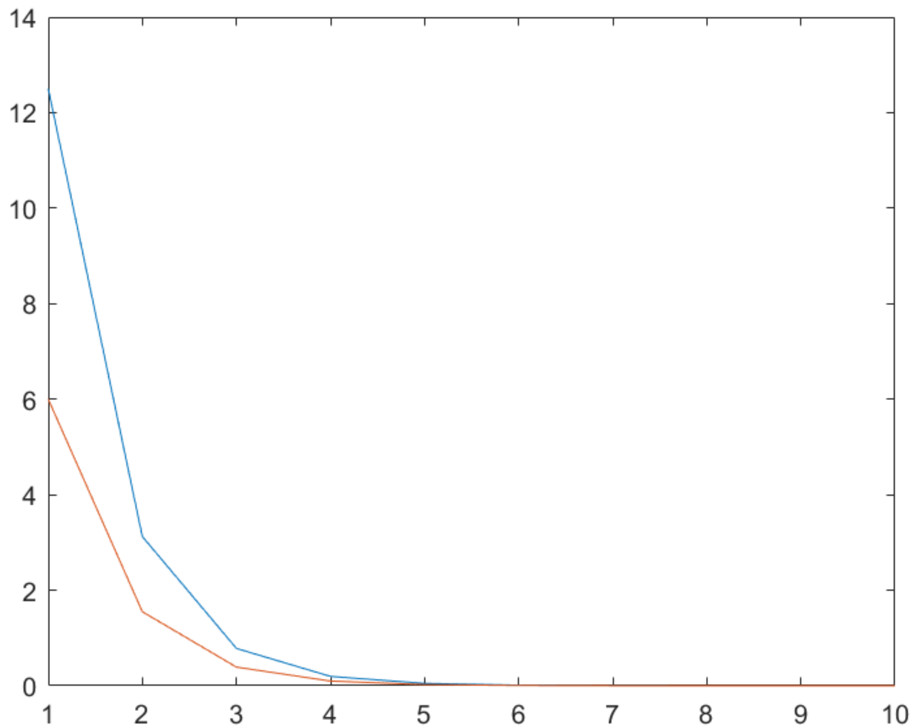
If no code shown 0
1 mark for having the 2 lines
1 mark for correct answer for both

f)From b) Both eigenvectors are less than 1 therefore you expect the solution to go to the fixed point (0,0).
From c) the limit as $t- > \infty$ of the exact solution is (0,0). since b, s_E are rates hence less than 1
From d) the limit as $t- > \infty$ of the exact solution is (0,0)
From e) seems to be approaching (0,0)
Any of the above explanations work, 2 marks for having a good explanation.

g)

```
for t=1:10
    Adults(t)=C1*b^t*C(1,2)+C2*s_E^t*C(1,1);
    Elders(t)=C1*b^t*C(2,2)+C2*s_E^t*C(2,1);
    x(t)=t;
end
plot(x,Adults,x,Elders)
```

H



2 marks for the graph

1. QUESTION 2

Given the system $x' = x^2 - 2x - 3$.

a) Classify the model

b) Find the fixed points

c) Find the stability of the fixed points

d) Find an explicit solution of the model

For the following use computer software:

e) Run three numerical experiments with initial conditions of (X(0)=-5, X(0)=1, X(0)=10) and use the explicit formula with $t = 1, 10, 100$ to determine the long term dynamics of each initial condition.

f) Do your results in part e) agree with part b), c)? Explain!

21 marks for question 2
1 mark for a)
3 marks for b)
4 marks for c)
7 marks for d)
3 marks for e)
3 marks for f)
a) Nonlinear Univariate Continuous Deterministic (NUCD)
1 mark

b) $0 = x^2 - 2x - 3$
$0 = (x-3)(x+1)$
$x = 3, -1$
1 mark for left side = 0
1 mark for factoring
1 mark for getting -1 and 3

c)$f(x) = x^2 - 2x - 3$
$f'(x) = 2x - 2$
$f'(-1) = -4 < 0$ therefore -1 is a stable fixed point
$f'(3) = 4 > 0$ therefore 3 is an unstable fixed point
1 mark for f'
1 mark for sticking in the fixed points
1 mark for comparing with 0
1 mark for having both correct conclusions (not 1 mark each 1 total)
d) $\frac{dx}{(x+1)(x-3)} = dt$
Using partial fractions, the left hand side becomes
$\frac{1}{4}\frac{dx}{x-3} + \frac{-1}{4}\frac{dx}{x+1}$
$\int \frac{1}{x-3}dx - \int \frac{1}{x+1}dx = 4\int 1dt$
$ln(x-3) - ln(x+1) = 4t + C_1$
$ln(\frac{x-3}{x+1}) = 4t + C_1$
$\frac{x-3}{x+1} = C_2e^{4t}$
$x - 3 = C_2e^{4t}(x+1)$
$x(1 - C_2e^{4t}) = 3 + C_2e^{4t}$
$x(t) = \frac{3+C_2e^{4t}}{1-C_2e^{4t}}$
at $t = 0$, $x(t) = x(0) = \frac{3+C_2}{1-C_2}$
$x(0)(1 - C_2) = 3 + C_2$
$C_2(1 + x(0)) = x(0) - 3$
$C_2 = \frac{x(0)-3}{1+x(0)}$

1 mark for separation of variables 1st step
1 mark for partial fractions correctly
1 mark for integration correctly (correct signs included)
1 mark for using e corrected to get rid of the ln's
1 mark for rearranging and getting x(t) in terms of stuff with C_2
1 mark for attempting to solve C_2 in terms of x_0
1 further mark more for the restating x(t) in terms of x_0 with no more C_2

e)

```
for i=1:3
t=10^(i-1);
x(1)=-5;
(3+(x(1)-3)/(1+x(1))*exp(1)^(4*t))/(1-(x(1)-3)/(1+x(1))*exp(1)^(4*t))
x(1)=1;
(3+(x(1)-3)/(1+x(1))*exp(1)^(4*t))/(1-(x(1)-3)/(1+x(1))*exp(1)^(4*t))
x(1)=10;
(3+(x(1)-3)/(1+x(1))*exp(1)^(4*t))/(1-(x(1)-3)/(1+x(1))*exp(1)^(4*t))
end
```

-1.0370 -1 -1
-0.9281 -1 -1
-1.1185 -1 -1

-1.036969840905367 -1 -1
-0.928055160151634 -1 -1
-1.118538618123929 -1 -1

If no code 0 marks.
1 mark for some resemblance of code doing what I have above
1 mark for 1st column of output
1 mark for last 2 columns of output
Note the answer can be read as the first column being t=1, 2nd t=10, 3rd t=100

f) Yes
-1 is the stable fixed point and ALL solutions are attracted/approcing it
1 mark for sating 1
1 mark for Yes being true (they have -1 as their output for e)
1 mark for ALSO starting that all the solutions seem to be approaching

the fixed point -1