MATH 3MB3 Review

Midterm Date: Friday, October 5

1 Linear Univariate Discrete Deterministic

$$N(t+1) = f(N(t)) \tag{1}$$

- **Equilibria**: set $N(t) = N(t+1) = N_*$ in equation 1 and solve for N_*
- **Stability of equilibria**: stable if $|f'(N_*) < 1$, otherwise unstable
- **Solve for time-dependent solution**: refer to *Affine Models* or *hw1* for examples
- **Multiple Lags**:

  $$N(t+2) = aN(t+1) + bN(t)$$

  Let $N(t) = C\lambda^t$:

  $$\begin{aligned} C\lambda^{t+2} &= aC\lambda^{t+1} + bC\lambda^t \\ C\lambda^t(\lambda^2) &= C\lambda^t(a\lambda + b) \\ \lambda^2 &= a\lambda + b \\ \lambda^2 - a\lambda - b &= 0 \end{aligned}$$

  Solve for λ (using quadratic equation - you will get two values) and substitute it into the general solution of the homogeneous equation:

  $$N(t) = C_1\lambda_1^t + C_2\lambda_2^t$$

  Find C_1 and C_2 from initial values, i.e. $N(0)$ and $N(1)$.

2 Nonlinear Univariate Discrete Deterministic

$$N(t+1) = f(N(t)) \tag{2}$$

f in equation 4 is a *non-linear* function. However, all analysis is the same as a LUDD model, but will likely be more difficult.

3 Linear Multivariate Discrete Deterministic

$$\vec{x}(t+1) = A\vec{x}(t) \tag{3}$$

In equation 3, $\vec{x}(t) = (x_1(t), ..., x_n(t))$. The vector $\vec{x}(t)$ has n components and the matrix A is an n x n matrix.

- **Fixed points**: $\vec{x}(t) = A\vec{x}(t)$
- **Time-dependent solution**:
  1. **Direct approach**: $\vec{x}(t) = A^t\vec{x}(0)$
  2. **Eigenvalue approach**: $\vec{x}(t) = (SD^tS^{-1})\vec{x}(0)$, where S is a matrix whose columns are the eigenvectors of A and D is a matrix with the eigenvalues on its diagonal and zeroes everywhere else
- **Stability**: Dominant eigenvalue λ_d
  1. **Stable**: $|\lambda_d| < 1$
  2. **Unstable**: $|\lambda_d| > 1$
  3. **Interesting**: $|\lambda_d| = 1$

4 Nonlinear Multivariate Discrete Deterministic

$$\vec{x}(t) = \vec{f}\,\vec{x}(t) \tag{4}$$

$\vec{f}$ is a vector-valued function, which is non-linear.

- **Fixed points**: Look at $S = S_*$ and $I = I_*$ simultaneously
- **Stability**: Eigenvalues of the Jacobian determine stability (all eigenvalues must be less than one in order for equilibrium point to be stable)

5 Linear Univariate Continuous Deterministic

$$\frac{dx(t)}{dt} = rx(t) \tag{5}$$

- **Fixed point**: set equation 5 to 0
  - $r \neq 0$: fixed point is $x_* = 0$
  - $r = 0$: every real number is a fixed point
- **Stability**:
  - $x > 0$: stable if $r < 0$
  - $x < 0$: stable if $r > 0$
- **Explicit solution**: $x(t) = x(0)e^{rt}$
- **Affine models**: $\frac{dx}{dt} = a - bx$ gives an explicit solution of $x(t) = e^{-bt}(x(0) - \frac{a}{b}) + \frac{a}{b}$

6 Nonlinear Univariate Continuous Deterministic

$$\frac{dx(t)}{dt} = f(t, x(t)) \tag{6}$$

f in equation 6 is a nonlinear function.

- **Fixed points**: $f(x_*) = 0$
- **Stability**: Stable if $f'(x_*) < 0$
- **General solution**: done by solving ODE