What we are now interested in is where different parameters or some part of the model is allowed to have it's value change over time, and this change depends on some chance/probability. To start with we are first going to review basic probability before looking at actual models.

1. Probability Introduction

Lets start with two events A, B. Then the probability of either A or B occuring/happening is:
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Where the last term is denoted as the joint probability of A and B, which means both A and B have occurred.

If two events are deemed to be *Mutually Exclusive*, that means that the probability of them both occurring at the same time is 0, so our above equation would simplify to $P(A \cup B) = P(A) + P(B)$
A simple example of mutually exclusive events, is take a coin and let A be the event where Heads comes up (face up) after flipping the coin. Let B be the event where Tails comes up after flipping the coin. If you only flip the coin once, you can't have both heads and tails come up at the same time.

The sum of the probabilities of all possibly outcomes of a set of events is always equal to 1. This often becomes useful if one of the events is hard to find/calculate, you can simply take 1 and subtract off all of the other events to find that event.

The conditional probability of A given B is written as P(A—B) (which reads as Probability of A such that B has occurred) is:
$P(A|B) = \frac{P(A \cap B)}{B}$
The formula for the unconditional probability of A is:
$P(A) = P(A|B) + P(A|notB)$
Which simply states that the probability of A occurring is simply the probability of A occurring given B has occurred plus the probability of A occurring given that B has not occurred.
If $P(A) = P(A|B)$, then this can be read as the probability of A is equal to the probability of A given B has occurred. Which means that

the probability of A occurring has nothing to do with B, meaning A is independent of B. Another way to write this is that:
$P(A \cap B) = P(A)P(B)$

A *Distribution* is a law that defines the chance/odds/probability of a certain event occurring. A $Discrete - Distribution$ is defined by a $probabilitydistributionp(x) = P(X = x)$. This last part reads as the probability of the random variable X being the value of x. For a continuous distribution, a *probability density function* $F(x)$ is used instead and it is defined as follows:
$f(x) = \lim_{\Delta x \to 0} \frac{P(x<X<x+\Delta x)}{\Delta x}$
Alternatively, one can use the *cummulative distribution function* $F(x)$, with $F(x) = P(X \leq x)$ for a discrete random variable X and
$F(x) = \int_{-\infty}^{x} f(x)dx$
for a continuous variable. This function F(x) is often abbreviated as CDF.

An important property of a distribution are its *moments*. The first moment is the *mean* $\mu = \bar{x}$, also called the expected value, denoted as E(X), of the random variable X, given by:
$E(X) = \sum_x xP(x)$
$E(X) = \int xf(x)dx$
For the discrete and continuous case respectively. Note that the expected value can be treated as a linear operator. Meaning that $E(aX + bY) = E(aX) + E(bY) = aE(X) + bE(Y)$. However since it is a linear operator, $E(f(X)) \neq f(E(X))$ if f is a nonlinear function.
The second moment is the *variance* denoted by $\gamma = V(X)$ and definite as:
$V(X) = E((X - E(X))^2)$
which is equivalent to
$V(X) = E(X^2) - E(X)^2$

2. DISCRETE EXAMPLE

A linear univariate discrete stochastic model can be written as:
$N(t + 1) = (1 + r)N(t)$
Where N is our state variable, t is time, and r is a random parameter. This means that at each time, r is chosen from a distribution law or function. Note since r is random, this will also cause N to be random as well.

Assume that the population grows at a rate of r_G in a good year and declines at a rate of r_B in a bad year, so at a minimum, $r_G > 0$ and $r_B < 0$ and r is either r_G or r_B depending if it is a good or bad year. We also have to know the rates/odds/probability of a good and bad year occurring. Let p denote the probability of a good year, and let q be the probability of denoting a bad year. Note that $0 \leq p \leq 1$, $0 \leq q \leq 1$, and $p+q = 1$. So r is a Bernoulli variable that takes on the values of r_g with probability p, and r_b with probability $q = 1-p$. Furthermore it is assume that the probability of a good year or a bad year have nothing to do with what happened in the previous year. Therefore p and q are independent.

3. Finding the distribution

If we start the previous example with a population of $N(0)$, then we infer that $N(1) = (1 + r_G)N(0)$ with probability p and $N(1) = (1+r_B)N(0)$ with probability $1-p$. Furthermore for N(2) we have the following:
$P(N(2) = (1 + r_G)^2N(0)) = p^2$
$P(N(2) = (1 + r_B)^2N(0)) = q^2$
$P(N(2) = (1 + r_G)(1 + r_B)N(0) =?$ Well we know that $p + q = 1$ therefore if we square both sides we get $p^2 + 2pq + q = 1$. Since the sum of all probabilities at a certain time must add up to one the remaining event happens with a probability of $2pq$. It is easily seen that as we go further and further out in time we are going to get more and more equations. We need another way of looking at this. What if we wanted to find the probability that after t years, we have had n good years, and t-n bad years. We get the following:
$P(N(t) = (1 + r_G)^n(1 + r_B)^{t-n}N(0)) = \binom{t}{n}p^nq^{t-n}$
$= \frac{t!}{(t-n)!n!}p^nq^{t-n}$
This defines a binomial distribution.

It is useful to calculate the expected value of N(t). The binomial distribution we are looking at has a t year time period, with p chance of good, gives us an expected value of pt years of good, and similarly qt years of bad. A direct consequence of this is that the expected value is simply:
$E(N(t)) = (1 + r_G)^{pt}(1 + r_B)^{(1-p)t}N(0)$

4. Fixed points and stability

Fixed points are found in the same way as for deterministic models: by setting $N(t) = N(t+1) = N^*$ and solving for N^*. Keep in mind though that the parameters in the problem are no longer constant! The example above is exceptional in its simplicity: the only fixed point is $N^* = 0$.

With regards to stability, the conclusions from deterministic models hold but there is usually less room to make specific conclusions because of the presence of stochasticity. In the example looked at above, stability is clearly guaranteed if 1 + rG and 1 + rB are both less than 1. If either is greater than 1, the state variable N(t) may grow in some years. It is then necessary to determine if this growth is sufficiently large and frequent as to offset the decrease in bad years. To do this, we examine the expected value of N(t), which can be rewritten as
$E(N(t)) = ((1 + r_G)^p(1 + r_B)^{(1-p)})^t N(0)$
It now follows that for stability we simply need $(1+r_G)^p(1+r_B)^{(1-p)} < 1$. There are two issues. First, there is the fact that the condition only guarantees that the system approaches the fixed point, but not necessarily that the convergence is monotonic. Second, it only guarantees that the expected value approaches the fixed point. With some bad luck, the dynamics exhibited by a single trial can differ significantly from the one predicted by the expected value. The expected deviation from the mean is quantified by the variance (which therefore is a useful quantity to compute).

5. Numerical results

First lets pick some values. Let $p = 0.2, q = 0.8, r_G = 0.3$ $r_B = -0.1$, and $N(0) = 13$. There are a bunch of different ways to look at how to simulate this is Matlab. Since what you have may be much more complicated, what you can do is something like the following:

```
p=0.2;
q=0.8;
rG=0.3;
rB=-0.1;
N(1)=13;
rng(1,'twister'); %This generates a seed value for a random number generator.
% What this means is if I pick the seed of 1, and ask for 10
% random numbers. I can then come back later, again have this
% line of code the asking for 10 random numbers will give me
```

```
% the exact same random numbers.  This basically helps and saves reproducibilit
count(1)=1; %counter variable for plotting later on
for i=1:80
    r1=rand; %This sets r1 to be a random value between 0 and 1
    count(i+1)=i+1;
    if r1>0.2
        N(i+1)=(1+rB)*N(i)
    else
        N(i+1)=(1+rG)*N(i)
    end
end
plot(count-1,N) %note this -1 is to keep the initial condition in the plot
```

If instead you wanted to do a bunch of simulations and to see what the expected values are there would be a slight change

```
p=0.2;
q=0.8;
rG=0.3;
rB=-0.1;
simulations=5; %pick number of simulations
N(1,1:simulations)=13; %This now sets all simulation run initial conditions
rng(1,'twister'); %This generates a seed value for a random number generator.
% What this means is if I pick the seed of 1, and ask for 10
% random numbers. I can then come back later, again have this
% line of code the asking for 10 random numbers will give me
% the exact same random numbers.  This basically helps and saves reproducibilit
count(1)=1; %counter variable for plotting later on
for j=1:simulations
    for i=1:80
        r1=rand; %This sets r1 to be a random value between 0 and 1
        count(i+1)=i+1;
        if r1>0.2
            N(i+1,j)=(1+rB)*N(i,j);
        else
            N(i+1,j)=(1+rG)*N(i,j);
        end
    end
end
E=sum(N.')/j %.' transposes the matrix, and sum takes the sum of the columns
% So this is one way to get a row sum out
plot(count-1,N,count-1,E) %does all the simulations and also puts the mean on t
```

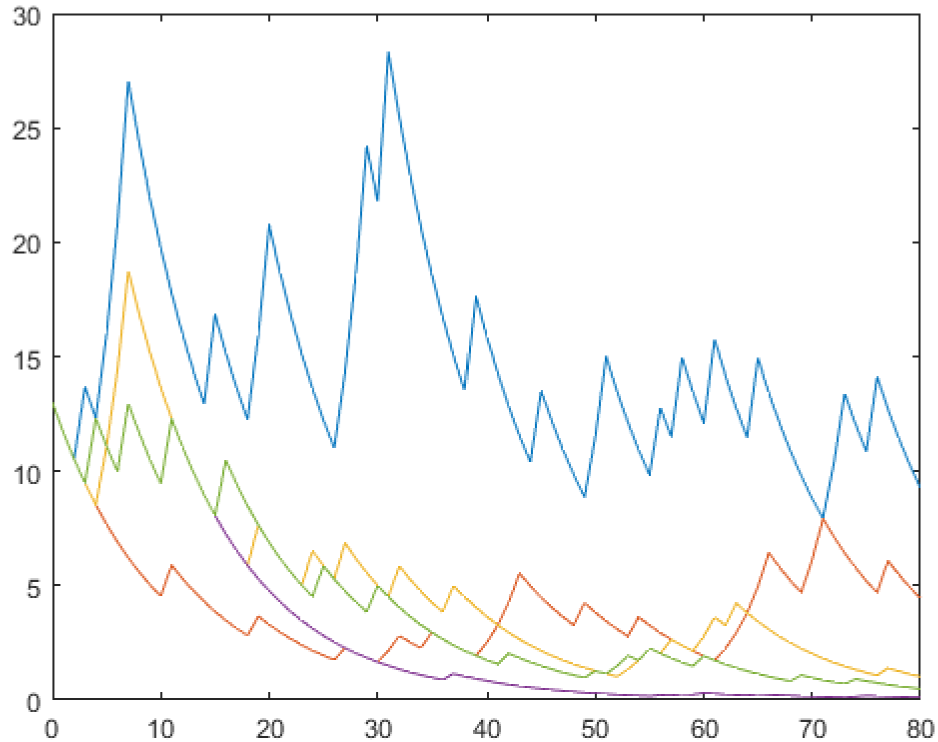

Figure 1. Code from above this shows all 5 simulations

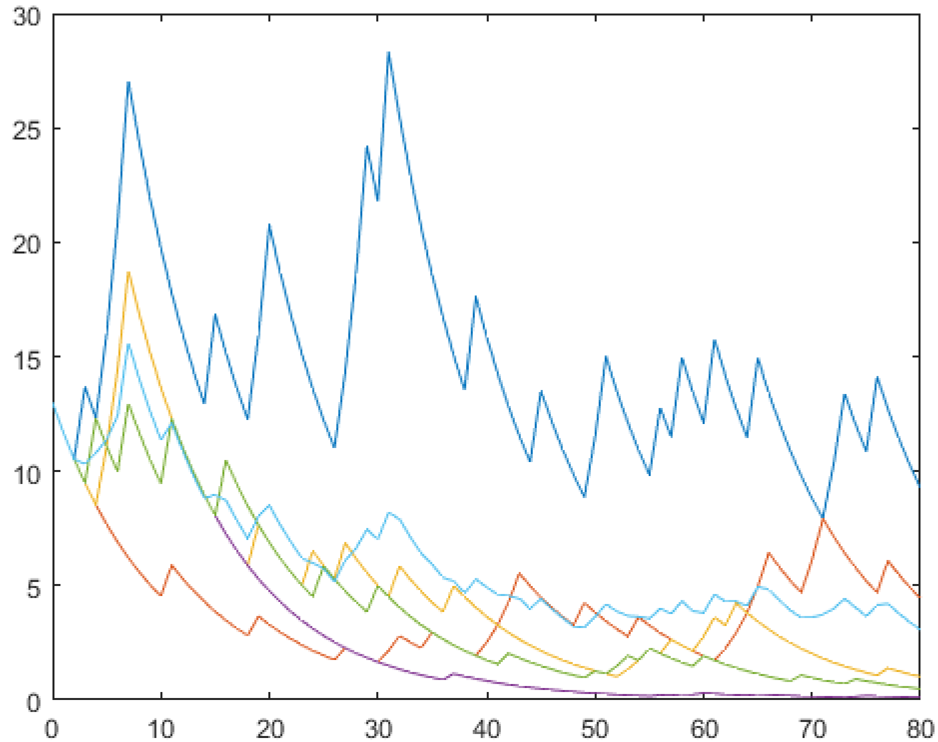

Figure 2. Same as the above plot except this also has the expected value on the graph as well