One type of multivariate stochastic model is the *Markov chain*. Although these discrete models are reminiscent of certain deterministic models, in the sense that we will define them by a simple matrix equation, an important difference is their interpretation in the framework of chance and probability. A Markov chain is a multivariate linear model of the form
$\vec{x_{t+1}} = T\vec{x_t}$
where T is an n x n matrix, t is time, and $\vec{x}$ is a vector of length n.

1. Introduction

Each component of the state vector $\vec{x}$ is associated with a certain state of the system. The numerical values of the n components of $\vec{x_t}$ reflect the probability of the system being in that particular state at time t. We thus require the components of $\vec{x_t}$ to lie between 0 and 1 and for the components to sum to 1. Additionally, the matrix

$$\begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix}$$

must have the columns adding up to 1. Which means that for each i:
$\sum_{j=1}^{n} p_{ij=1}$
The matrix T is called the *transition matrix* and its entries p_{ij} represent the (conditional) probability of the system transitioning from state i to state j. Note that for all the Markov chain matrices, it is assumed that each move or each transition matrix does not change/vary based off the the previous moves. This means that each move is independent of past moves. This is called the *Markov Property*.

2. Explicit Solution

As before we have:
$\vec{x_t} = T^t\vec{x_0}$
Where $\vec{x_0}$ is the given initial condition. The catch is the calculation of T^t. Note that the power t will be used many times throughout these notes, and never means the transpose. The eigenvalue approach, that we outlined before, still works.

3. Fixed Points

Because of the properties of the transition matrix, it is impossible to have an eigenvalue larger than 1. Furthermore the dominate eigenvalue will always be 1. Because of this, the eigenvector(s) corresponding to the eigenvalue of 1, is known as the stable *equilibrium distribution* of T. It is also called the $Perron - Frobenius\ eigenvector$.

4. Asymptotics

The limit as $t \to \infty$ is of particular interest, as it dictates the convergence of the initial condition $\vec{x_0}$ after a long period of time. In this case what we are looking for is:
$T^t \to L$ as $t \to \infty$
where L is an n x n matrix. This matrix is called the $steady - state$ $matrix$. All of the entries are real numbers, and $L^2 = L$, meaning that L is *idempotent*. Equilibrium states are necessarily eigenvectors of L with eigenvalue of 1.

5. Absorbing states

A Markov Chain has *absorbing states* if the matrix T can be rearranged into the following form:

$$\begin{bmatrix} A & 0 \\ B & I \end{bmatrix}$$

where A and B are matrices, 0 is a zero matrix, and I is the identity matrix. The rearranging is achieved by relabelling of the states

Lets say that A is an $(n-k)x(n-k)$ submatrix, which then makes I the kxk identity matrix. Then the components associated with the final k components of the state vector, are all absorbing states. They are left invariant in time by T and absorb other states of the system through multiplication with the submatrix B.
One can show for a matrix T of the block form that:

$$T^t = \begin{bmatrix} A^t & 0 \\ B(I-A^t)(I-A)^{-1} & I \end{bmatrix}$$

so the time-dependent solution is then:

$$\vec{x_t} = \begin{bmatrix} A^t & 0 \\ B(I-A^t)(I-A)^{-1} & I \end{bmatrix} \vec{x_0}$$

Furthermore because of the absorbing states we know that $A^t \to 0$ as $t \to \infty$ therefore:

$$\vec{x_t} = \begin{bmatrix} 0 & 0 \\ B(I-A)^{-1} & I \end{bmatrix} \vec{x_0} = L\vec{x_0}$$

is the fixed point for the absorbing Markov Chain.

Note that every absorbing state represents an equilibrium but not vice versa. The matrix T may be free of zeros, which excludes the presence of absorbing states. However, T will always have 1 as an eigenvalue which implies it will always have an equilibrium.

6. Example with absorption

Say we are modelling the population size of a certain species. The population has three states: extinction (state 1), contraction (state 2), expansion (state 3). At t = 0, the probability that the population is in these three states is dictated by $\vec{x_0} = (0.01, 0.14, 0.85)$. The matrix governing state transitions is assumed to be

$$T = \begin{bmatrix} 1 & 0.05 & 0.01 \\ 0 & 0.7 & 0.19 \\ 0 & 0.25 & 0.8 \end{bmatrix}$$

First few things to note. The sum of all columns do add up to exactly 1. The first column reveals that once a population becomes extinct, it stays extinct, so extinction is an absorbing state. The second column describes the chance of transitioning from contracting to extinct ($p_{21} = 0.05$), remaining in contraction ($p_{22} = 0.7$) and contracting to expanding ($p_{23} = 0.25$). The last column similarly defines these conditional probabilities in the case of an expanding population.

We wish to write the matrix T into block form. To this end, we relabel the states to be expansion (state 1), contraction (state 2), extinction (state 3). Then we have $\vec{x_0} = (0.85, 0.14, 0.01)$ and:

$$T = \begin{bmatrix} 0.8 & .25 & 0 \\ .19 & 0.7 & 0 \\ 0.01 & 0.05 & 1 \end{bmatrix}$$

So $T = \begin{bmatrix} A & 0 \\ B & I \end{bmatrix}$

with

$$A = \begin{bmatrix} 0.8 & 0.25 \\ .19 & .7 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.01 & 0.05 \end{bmatrix}$$

The extinct state is therefore an absorbing state of the system. The steady-state matrix can be computed by calculating $B(I-A)^{-1} = (1, 1)$ Therefore

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and therefore $x^* = (0, 0, 1)$. It is clear to see that no matter the initial condition, eventually the species all dies out and end up with extinction.

7. Example without absorption

Now lets modify that previous example slightly

$$T = \begin{bmatrix} 0.8 & .25 & 0.001 \\ .19 & 0.7 & .01 \\ 0.01 & 0.05 & .989 \end{bmatrix}$$

We note that this matrix cannot be transformed into the earlier block form especially since it has no zero entries. The eigenvalues of T are: 1, .09631, 0.5259. The first eigenvalue is the one that corresponds to the equilibrium, while the later two vanish over time. The eigenvector corresponding to the eigenvalue of 1 is (0.1601, 0.1252, 0.7147), which is the equilibrium state.

Furthermore if you use computer software to try to evaluate T^t as $t \to \infty$, you end up with all the columns looking like the eigenvector corresponding to the eigenvalue of 1.