MATH 3MB3 FALL 2018 LINEAR MULTIVARIATE DISCRETE DETERMINISTIC

The basic model is now $\vec{x}(t+1) = A\vec{x}(t)$. Here $\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is a vector with n components, so that we have n variables to solve for, and A is an n by n matrix.

1. Population Example

As an example, lets let J be the number of juveniles in a population and A be the number of adults. We assume that the birth rate for juveniles is 0, and the birth rate for adults is f. We assume that the survival rate for juveniles is s_J and the survival rate for adults is s_A. We also assume that on surviving one year, juveniles become adults. This is then a coupled equation (since A and J depend on each other and can be written a few different ways:
$J(t+1) = fA(t)$
$A(t+1) = s_J J(t) + s_A A(t)$
or
$$\begin{bmatrix} J(t+1) \\ A(t+1) \end{bmatrix} = \begin{bmatrix} 0 & f \\ s_J & s_A \end{bmatrix} \begin{bmatrix} J(t) \\ A(t) \end{bmatrix}$$
or you can combined the two equations to get
$A(t+1) = s_J f A(t-1) + s_A A(t)$

2. fixed point(s)

We say that $\vec{x}$ is a fixed point if $\vec{x} = A\vec{x}$, or equivalently $0 = (A-I)\vec{x}$, where I is the identity matrix that is the same size as the square matrix A. The null space of $A - I$ therefore contains all the fixed points. If $A - I$ is invertible, we only have $\vec{x}^* = 0$. If $A - I$ is not invertible, there is an $n - r$ dimensional space of fixed points, r being the rank of that matrix. Remember that a matrix is invertible if its determinant is not zero. For the juvenile-adult model, the determinant of $A - I$ is $1 - s_A - f s_J$. This quantity may be zero depending on the value of the constants f, s_A and s_J . In what follows, we assume it is not zero so that the only fixed point is $A^* = 0 = J^*$

3. Time-dependent solution

There are two different approaches to finding the solution of a model of this form. 1) Directly attacking the problem. 2) Using the eigenvalues of the matrix A and some linear algebra.

4. The direct approach

Like with out earlier models, try to apply the recursion a few times and see if you can find a pattern.
$\vec{x}(1) = A\vec{x}(0)$
$\vec{x}(2) = A\vec{x}(1) = A^2\vec{x}(0)$
$\vec{x}(3) = A\vec{x}(2) = A^2\vec{x}(1) = A^3\vec{x}(0)$
Therefore:
$\vec{x}(t) = A^t\vec{x}(0)$
Note that actually figuring out what A^t is in a question can be tricky however if a computer is available it is pretty easy to figure out.

5. eigenvalue approach

So a better way to process is to diagonalize the matrix A, which can be done by writing it as SDS^{-1}, where S is an n by n matrix whose columns are the eigenvectors of A and D is a matrix with the eigenvalues of A on its diagonal and zeroes everywhere else. Subbing this into the matrix power equation one finds.
$\vec{x}(t) = A^t\vec{x}(0) = (SDS^{-1})^t\vec{x}(0)$
$=(SDS^{-1})(SDS^{-1})\dots(SDS^{-1})\vec{x}(0)$
$=(SD^tS^{-1})\vec{x}(0)$
The term D^t is quite simple to find since D is diagonal, hence the power of t is applied to the diagonal elements of D. For example:
D=$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ $D^t = \begin{bmatrix} d_1^t & 0 \\ 0 & d_2^t \end{bmatrix}$

6. change of variables

An even neater trick is to view the pervious equation as $S^{-1}\vec{x}(t) = D^tS^{-1}\vec{x}(0)$ and let $\vec{y}(t) = S^{-1}\vec{x}(t)$. Then the equation turns into $\vec{y}(t) = D^t\vec{y}(0)$, which is the same as:
$y_1(t) = d_1^t y_1(0)$
$y_2(t) = d_2^t y_2(0)$
$\vdots$
$y_n(t) = d_n^t y_n(0)$
Now the we have to solution we just have to go back to solving $\vec{x}(t)$ which is done by setting $\vec{x}(t) = S\vec{y}(t)$.

7. stability

The real numbers $y_1(t), y_2(t), \dots, y_n(t)$ are the coordinates of the vector $\vec{x}(t)$ in the basis of eigenvectors $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ of A, meaning that $\vec{x}(t) = y_1(0)d_1^t\vec{v_1} + y_2(0)d_2^t\vec{v_2} + \dots + y_n(0)d_n^t\vec{v_n}$

This expression reveals the importance of the dominant eigenvalue (i.e. eigenvalue with largest absolute value): all terms except for the dominant become negligible as time increases because all the eigenvalues d_i appear as d_i^t

The dominant eigenvalue consequently determines stability: if the absolute value of the dominant eigenvalue is greater than 1, a fixed point is unstable. If it is smaller than 1, it is stable. Things become interesting when the absolute value of the dominant eigenvalue equals 1, or when it appears twice.

8. Back to the population example

We are going to do this by the eigenvalue approach which requires us to solve the equation $det(A - DI) = 0$. For this model using the quadratic formula we get
$d_t = \frac{s_A \pm \sqrt{s_A^2 + 4s_j f}}{2}$
Which then requires us to solve the following:
$\begin{bmatrix} 0 & f \\ s_J & s_A \end{bmatrix} \begin{bmatrix} v_{2t-1} \\ v_{2t} \end{bmatrix} = d_t \begin{bmatrix} v_{2t-1} \\ v_{2t} \end{bmatrix}$
to find the eigenvectors. For the dominant eigenvalue, we get:
$f v_2 = \frac{s_A + \sqrt{s_A^2 + 4s_j f}}{2} v_1$
$s_J v_1 + s_A v_2 = \frac{s_A + \sqrt{s_A^2 + 4s_j f}}{2} v_2$
and for the non dominant eigenvalue we get:
$f v_4 = \frac{s_A - \sqrt{s_A^2 + 4s_j f}}{2} v_3$
$s_J v_3 + s_A v_2 = \frac{s_A - \sqrt{s_A^2 + 4s_j f}}{2} v_4$
Using the shortcut from the section above we can also conclude that:
$\vec{x}(t) = c_1 d_1^t \vec{v_1} + c_2 d_2^t \vec{v_2}$
Where c_1 and c_2 can be determined from the initial condition $\vec{x}(0)$, and $\vec{v_1} = (v_1, v_2)$ and $\vec{v_2} = (v_3, v_4)$.

9. affine model

The multivariate affine model has the form $\vec{x}(t+1) = A\vec{x}(t) + \vec{b}$. For fixed points solve $\vec{x}^* = A\vec{x}^* + \vec{b}$. If $A - I$ is invertible, then the solution is $\vec{x}^* = -(A - I)^{-1}\vec{b}$. The stability conditions are the same as in the linear case described above.