The basic model is now
$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$
where $\vec{f}$ is a vector-valued function of a vector-valued state variable $\vec{x}$. As before we find fixed points by setting $\frac{d\vec{x}}{dt}$=0 and solving, though depending on the model this can become tricky.

1. Fixed Points

$$\frac{dM}{dt} = aM(1-M)(N-\frac{1}{2})$$
$$\frac{dN}{dt} = bN(1-N)(M-\frac{1}{2})$$

So first we replace the left hand side by 0's.

$$0 = aM(1-M)(N-\frac{1}{2})$$
$$0 = bN(1-N)(M-\frac{1}{2})$$

Now we find solutions to the first equation ($M = 0, M = 1, N = \frac{1}{2}$) and place each of those into the second equation to get the following:
$0 = bN(1-N)(-\frac{1}{2})$
$0 = bN(1-N)(\frac{1}{2})$
$0 = \frac{b}{2}(\frac{1}{2})(M-\frac{1}{2})$
Which gives us the following set of fixed points:
$(0,0), (0,1), (1,0), (1,1), (\frac{1}{2}, \frac{1}{2})$
Note it is possible for a nonlinear model to have infinitely many fixed points, or no fixed points.

2. Stability

Like we have seen before, a fixed point is stable iff the real part of all of the eigenvalues are less than 0. Looking at the example above our Jacobian matrix looks like:

J= $\begin{bmatrix} a(N-\frac{1}{2})-2aM(N-\frac{1}{2}) & aM(1-M) \\ bN(1-N) & b(M-\frac{1}{2})-2bN(M-\frac{1}{2}) \end{bmatrix}$

Which can be simplified to:

$\begin{bmatrix} a(N-\frac{1}{2})(1-2M) & aM(1-M) \\ bN(1-N) & b(1-2N)(M-\frac{1}{2}) \end{bmatrix}$

Now lets look at this Jacobian under each of the sets of fixed points

J(0,0)= $\begin{bmatrix} \frac{-a}{2} & 0 \\ 0 & \frac{-b}{2} \end{bmatrix}$

This gives us eigenvalues of $\frac{-a}{2}$ and $\frac{-b}{2}$. The origin/fixed point will be stable if both a and b are positive.

J(1,0)= $\begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{b}{2} \end{bmatrix}$

This gives us eigenvalues of $\frac{a}{2}$ and $\frac{b}{2}$. (1,0) will be stable if both a and b are negative.

J(0,1)= $\begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{b}{2} \end{bmatrix}$

This gives us eigenvalues of $\frac{a}{2}$ and $\frac{b}{2}$. (0,1) will be stable if both a and b are negative.

Note in this case it is allowable to have different fixed points stable at the same time as there will exist a separatrix (dividing region) under which certain initial conditions will go to one while certain other initial conditions will go to the other fixed point.

$J(\frac{1}{2},\frac{1}{2}) = \begin{bmatrix} 0 & \frac{a}{4} \\ \frac{b}{4} & 0 \end{bmatrix}$

Note we can't directly pull off the eigenvalues for this one therefore we will have to calculate them. The characteristic polynomial that we get will be $\lambda^2 - \frac{ab}{16} = 0$ which gives us eigenvalues of $\pm\frac{\sqrt{ab}}{4}$. Note in this case if both a and b have the same sign, then these eigenvalues will have opposite signs, and the fixed point will be unstable. In the case where a and b have difference signs, then we end up with complex eigenvalues that have a zero real part, and stability has to be analyzed by a computer.

3. Time dependent solution

The results of a numerical simulation of the solution close to the fixed point $M^* = N^* = \frac{1}{2}$ are displayed in Figure 1. In close proximity to the fixed point, the nonlinear problem is well approximated by the linearization (i.e. the Jacobian). Further away from the fixed point, the shape of the orbit in the figure can not be predicted from the Jacobian; it is due to nonlinear effects.

4. CLASSICAL LOTKA-VOLTERRA MODEL

The Lotka-Volterra is probably the most well known example of a nonlinear multivariate continuous model. It describes the evolution of two populations in time: x represents prey and y predators. The system governing x(t) and y(t) is:

$$\frac{dx}{dt} = ax - bxy = x(a - by)$$
$$\frac{dy}{dt} = cxy - dy = y(cx - d)$$

where a,b,c,d are all positive constants. There are two fixed points for this system. An extinction equilibria at (0,0) and a coexistence equilibria at $(\frac{d}{c}, \frac{b}{a})$. The Jacobian for this system is:

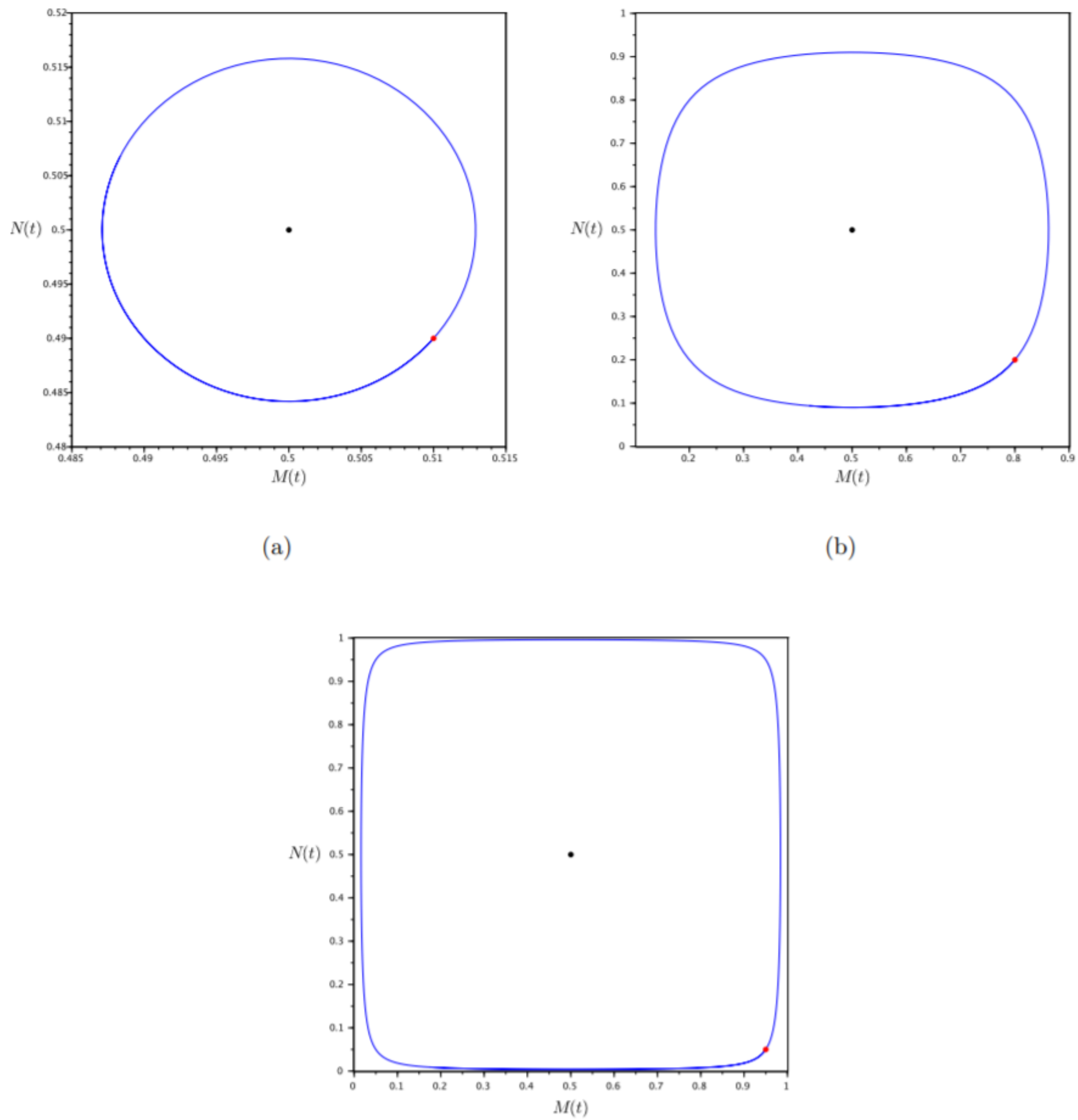
J= $\begin{bmatrix} a - by & -bx \\ cy & cx - d \end{bmatrix}$

J(0,0)= $\begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$

giving us eigenvalues of a and $-d$. Since all the parameters are positive constants $a > 0$, therefore this fixed point will be unstable. Since this is 2 dimensional what is happening is that the fixed point is unstable along the x axis (eigenvalue of a) and stable along the y axis (eigenvalue of -d). This type of fixed point is called a saddle point.

$J(\frac{d}{c}, \frac{b}{a}) = \begin{bmatrix} 0 & \frac{-bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}$

The eigenvalues of this matrix are $\pm i\sqrt{ad}$, both purely imaginary, meaning that we expect orbits close to the fixed point to be circular. For initial conditions further away from the equilibria we still expect orbits to be circular, however due to the nonlinearity, they may be deformed into ellipses for example. An example of this is displayed in figure 2.



(c)

FIGURE 1. The example is solved numerically for a=2, b=-3, in order to understand the coexistence equilibria $(\frac{1}{2}, \frac{1}{2})$. The equilibria is marked with a black dot, and the initial condition with a red dot. The initial condition for these three parts are: a) (0.51,0.49) b) (0.8,0.2) c) (0.95,0.05)

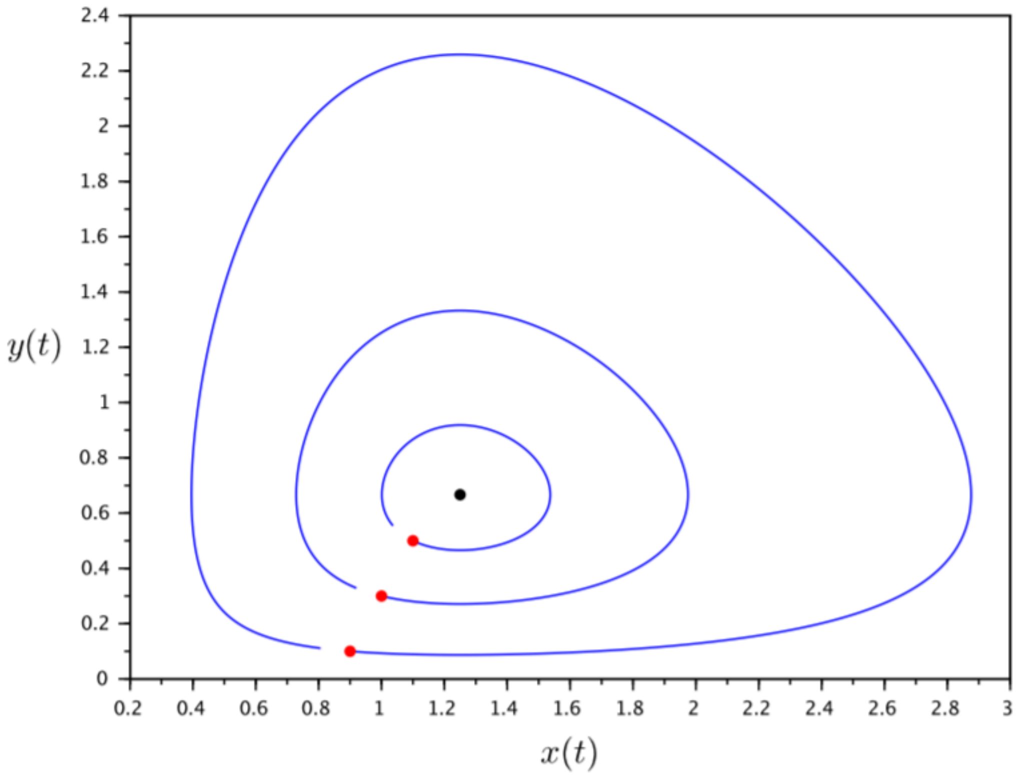


FIGURE 2. The Lotka-Volterra model is solved numerically, for a=2, b=3, c=4, d=5. The equilibria is again marked with a black dot, and the initial conditions are marked with a red dot. To show the direction of these orbits, the numerical simulation was halted before the orbit completed, to show the counterclockwise direciton