MATH 3MB3 FALL 2018 NONLINEAR MULTIVARIATE DISCRETE DETERMINISTIC

The basic model is now $\vec{x}(t+1) = \vec{f}\vec{x}(t)$. What is now new is that $\vec{f}$ is a vector-valued function of a vector-valued state variable $\vec{x}$, where t takes of discrete values. In two dimensions this looks like:
$\vec{f}(\vec{x}) = (\vec{f_1}(\vec{x}), \vec{f_2}(\vec{x}))$
where
$\vec{x}(t) = (\vec{x_1}, \vec{x_2})$

1. Epidemiological example

We start by looking at a disease which infects susceptible people exponentially with a contact rate of β (new infections per susceptible per infective per time period). The birth rate of new susceptible people is b, and the disease induced mortality is δ (extra chance of dying each time period, note this model doesn't account for base mortality (death not due to disease, so this becomes the only mortality), a recovery rate of γ(which means someone spends an average of $\frac{1}{\gamma}$ time in the infected class before recovering).
$S(t+1) = S(t) + bS(t) + \gamma I(t) - S(t)(1 - e^{-\beta I(t)})$
$I(t+1) = I(t) - \gamma I(t) - \delta I + S(t)(1 - e^{-\beta I(t)})$
Note this is clearly a non linear model due to the exponential function. If we want to look at the total population $N(t) = S(t) + I(t)$ then we end up with the equation:
$N(t+1) = N(t) + bS(t) - \delta I(t)$
Note in this case the total population is not constant, unless $b = \delta = 0$.

2. Fixed points

In this case we look at $S = S^*$ and $I = I^*$ simultaneously. Which gives us the following equations:
$0 = bS^* + \gamma I^* - S^*(1 - e^{-\beta I^*})$
$0 = -\gamma I^* - \delta I^* + S^*(1 - e^{-\beta I^*})$
adding these two together gives us:
$0 = bS^* - \delta I^*$
Therefore $I^* = \frac{bS^*}{\delta}$
Which when substituted into the first equation yields:
$0 = S^*(b + \frac{b\gamma}{\delta} - (1 - e^{-\frac{b\beta S^*}{\delta}}))$
meaning our fixed point is either $(0, 0)$ or:

$S^* = \frac{\delta}{\beta b} ln(\frac{1}{1-b-\frac{\gamma b}{\delta}})$
$I^* = \frac{1}{\beta} ln(\frac{1}{1-b-\frac{\gamma b}{\delta}})$
Therefore we have both the trivial equilibrium (0,0) where both populations have died out, and a nonzero non-trivial coexistence equilibrium (which may or may not be positive).

3. STABILITY

The stability of fixed points is investigated using the Jacobian matrix J of $\vec{f}$, which for the general 2 by 2 case equals

$$J(\vec{x}) = \begin{pmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

The Jacobian is a multivariate analog of the first derivative of functions of a single variable. A fixed point $\vec{x}$ is stable if all of the eigenvalues of this matrix (evaluated at the fixed point) have absolute value less than one. If the opposite is true, then it is unstable. In this way, the dominant eigenvalue determines stability of the equilibrium.
For the nonlinear SI model, we have that:
$f_1 = S + bS + \gamma I - S(1 - e^{-\beta I})$
$f_2 = I + -\gamma I - \delta I + S(1 - e^{-\beta I})$
Therefore the Jacobian is:

$$J(\vec{x}) = \begin{pmatrix} b + e^{-\beta I} & \gamma - \beta S e^{-\beta I} \\ 1 - e^{-\beta I} & 1 - \gamma - \delta + \beta S e^{-\beta I} \end{pmatrix}$$

At the extinction equilibrium (0,0), this Jacobian reduces to $J(0,0) =$

$$\begin{pmatrix} b+1 & \gamma \\ 0 & 1 - \gamma - \delta \end{pmatrix}$$

Since this is an upper triangular matrix, the eigenvalues can be read off of the diagonal. Therefore the stability conditions for this equilibrium are:
$|b + 1| < 1$ which is the same as $-2 < b < 0$
and
$|1 - \gamma - \delta| < 1$ which is the same as $0 < \gamma + \delta < 2$.
Note since γ and δ are rates they are already assumed to be between 0 and 1, hence this second inequality is satisfied automatically.
In terms of the non trivial equilibrium point, you would first have to have
$1 - b - \frac{\gamma b}{\delta} < 1$ just to make sure that the equilibrium itself is positive,

but to analyse this equilibrium is too messy to be done by hand and you would use a computer to check the jacobian evaluated at that equilibrium.