MATH 3MB3 FALL 2018 NONLINEAR UNIVARIATE CONTINUOUS DETERMINISTIC

The basic model is now
$\frac{dx(t)}{dt} = f(t, x(t))$
where f is a nonlinear function. This equation is always a first order differential equations. It allows for a wide array of applications to be modelled. For example, $f(x) = rx(1 - \frac{x}{K})$ gives the logistic model in continuous time.

1. Fixed points and stability

We are still looking at $\frac{dx}{dt} = 0$ but now that evaluates to $f(x^*) = 0$. Depending on the question it may be impossible to analytically solve for the fixed point and may require computer assistance. The fixed point x^* is stable if $f'(x) < 0$, as that results in small perturbations return to the fixed point.

2. Logistic Model

Letting $f(x) = rx(1 - \frac{x}{K})$ then solving for the fixed points nets us $0 = rx^*(1 - \frac{x^*}{K})$ so the fixed point is either 0 or K. $f'(x^*) = r(1 - \frac{2x^*}{K})$. $f'(0) = r$, $f'(K) = -r$. Therefore if $r > 0$, then K is stable and 0 is unstable.

3. Constant Harvest Model

The model is defined by:
$f(x) = rx(1 - \frac{x}{K}) - h$
It is like the logistic equation we have already done however has this addition constant -h term. Here h is the harvest amount that gets taken away continuously at each time. Note for this model it is no longer as trivial to actually find the fixed points.
$0 = rx - \frac{rx^2}{K} - h = x^2 - KX + \frac{hK}{r}$
$\frac{K \pm \sqrt{K^2 - \frac{4hK}{r}}}{2}$
If instead we had scaled the variables as $X = \frac{x}{K}$ and $H = \frac{h}{rK}$ then we end up with
$\frac{1 \pm \sqrt{1-4H}}{2}$
which is much easier to analyze.

If $H = \frac{1}{4}$ then there is a single fixed point of $\frac{1}{2}$
If $H > \frac{1}{4}$ then there are no fixed points.
If $H < \frac{1}{4}$ then there are two fixed points.
For determining their stability we have $f'(X^*) = r - 2rX^* = r(1-2X^*)$, which means the stability will depend on the sign of r.

4. Time dependent solution

There are basically two different approaches for time dependent solutions. If you can find a general solution using differential equation methods, do so, otherwise simulation the dynamics using numerical methods.

5. General Solution

An ordinary differential equation generally has the form $F(t, x(t), x'(t), x''(t), ...) = 0$ where F can be a complicated function of its variables. Importantly, it may include higher derivatives of x(t). In this class we will be finding general solutions for first order problems for which f has the form $f(t, x)$. Many problems, such as the logistic differential equation, can be done with separation of variables.

$\frac{dx}{x(1-\frac{x}{K})} = rdt$

Using partial fractions, the left hand side becomes

$\frac{dx}{x} + \frac{dx}{K-x}$

$\int \frac{1}{x}dx + \int \frac{1}{K-x}dx = r\int 1dt$

$ln(x) - ln(x-K) = rt + C1$

$ln(\frac{x-k}{x}) = -rt + C2$

$\frac{x-k}{x} = C3e^{-rt}$

$x - k = C3e^{-rt}x$

$x(t) = \frac{k}{1-C3e^{-rt}}$

at $t = 0$, $x(t) = x(0) = \frac{k}{1-C3}$

$x(0)(1-C3) = k$

$x(0) - k = x(0)C3$

$C3 = \frac{x(0)-k}{x(0)} = 1 - \frac{k}{x(0)}$

$x(t) = \frac{k}{1-(1-\frac{k}{x(0)})e^{-rt}}$

6. Numerical Example

If we start with a conservation of energy law.

$B = B_cN_c + E_c\frac{dN_c}{dt}$

Where B is the rate of energy intake (food, nutrient, power), B_c is the rate of intake required for a single cell, N_c is the total number of cells,

E_c is the energy required in the production of a new cell.

If we now denote m, and m_c to respectively be the total mass and the mass of a single cell, and use the empirical relationship $B = am^b$, where a and b are constants, and b is about $\frac{3}{4}$, we can derive a new ODE as follows with $m = N_c m_c$:
$\frac{dm}{dt} = \frac{dN_c}{dt} m_c = \frac{Bm_c}{E_c} - \frac{B_c m_c N_c}{E_c} = \frac{am^b m_c}{E_c} - \frac{B_c m}{E_c}$
Note the first term can be denoted as the supply while the second term is the demand. You can think of this as the organism must have enough energy (supply>demand) to grow. Now to find the fixed points:
$0 = \frac{am_c}{B_c} m^b - m = m^b(\frac{am_c}{B_c} - m^{1-b})$
Fixed points of 0 and $(\frac{am_c}{B_c})^{\frac{1}{1-b}}$
An interesting thing to note is that both equilibrium values are independent of E_c

Note if we look at a change of variables where $u = m^{1-b}$
$\frac{du}{dt} = \frac{du}{dm}\frac{dm}{dt}$
$\frac{du}{dt} = \frac{1-b}{m^b} m^b (\frac{am_c}{E_c} - \frac{B_c u}{E_c})$
$= \frac{(1-b)B_c}{E_c}(\frac{am_c}{B_c} - u)$
$= \frac{(1-b)B_c}{E_c}(u^* - u)$
This is now a separable equation which can now be solved. At which point you can then convert you u(t) back into the want m(t).