MATH 3MB3 FALL 2018 HOMEWORK 3
Due Monday October 15 at 11:59PM

Format your report using some form of word processing software (Word, Latex, OpenOffice, ...), export it to a PDF file and submit it via email to:

- Alexandra Bushby, bushbya@mcmaster.ca if your last name starts with A-G or if you are submitting using R
- Robert White, whitere@mcmaster.ca if your last name starts with a H-Z, or you plan on using Python, and you don't plan on using R

together with a file containing the code you used for your computer simulations.
Make sure that your email has the proper subject line and information from the outline.

Question 1

This question involves and explores the relationship between discrete and continuous models. First we start with a continuous model
$\frac{dx}{dt} = rx$
Then we discretize this equation using the Backward Euler method:
$\frac{dx}{dt} \approx \frac{x(t)-x(t-h)}{h}$
where h is a parameter, one obtains:
$X(t+h) = \frac{X(t)}{1-rh}$
We now denote it as capital X just to differentiate from the x in the continuous model we had before. Assume that x(0)=X(0)=a
a) Show how the third equation is obtained from the first two equations.
b) Find the solution x(t) of the first equation
c) Find an explicit expression for X(t) from the last equation, assuming that $t = nh$ with n being an integer.

Now set $r = 1.6$ and $a = 2$. Answer the following questions/parts with the aid of computer software. The domain of time that we are interested in is t=0 to t=6.
d) Graph x(t). On the same plot, graph X(t) for h=0.01 and h=0.001. Comment on your observations. Give the values of all three quantities at t=6.

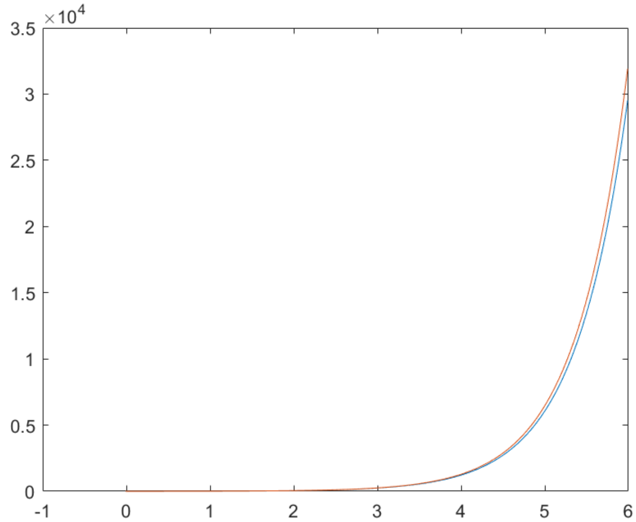
e) Graph the absolute global truncation error $E(t) = |x(t) - X(t)|$, again for both h=0.01 and h=0.001. What does E(t) represent? What is the value of E(6) for h=0.01? What is E(6) for h=0.001? Explain the difference between both values and speculate where the error comes from.
f) Graph the relative global truncation error $e(t) = \frac{|x(t)-X(t)|}{x(t)}$, again for both h=0.01 and h=0.001. What do you observe?

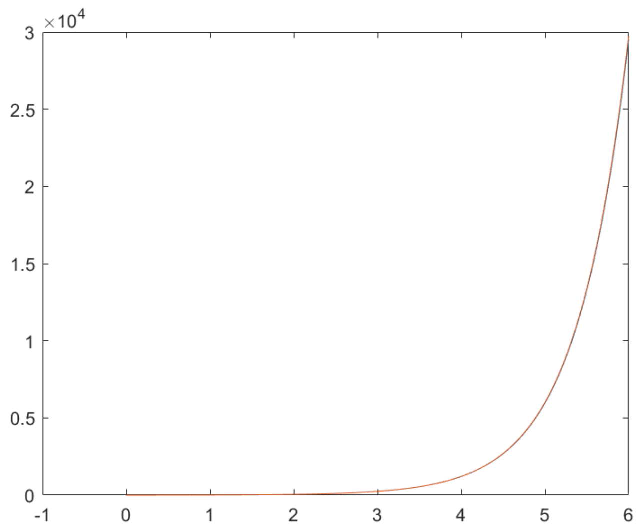
21 marks for question 1
3 for a)
2 for b)
3 for c)
4 for d)
6 for e)
3 for f)
a) $Let \frac{X(t)-X(t-h)}{h} = rX(t)$
$X(t) = hrX(t) + X(t-h)$
$X(t)(1-rh) = X(t-h)$
$X(t) = \frac{X(t-h)}{1-rh}$
$X(t+h) = \frac{X(t)}{1-rh}$

1 Mark for setting up the top line.
1 Mark for solving to the 2nd last line.
1 Mark for replacing t by t+h to get the final line.
b)$x(t) = x_0 e^{rt}$

2 marks, this has been done in class repeatedly, they can go directly to the answer. Should be no C's and just x_0.
c)$X(nh+h) = \frac{X(nh)}{1-rh}$
$X(h) = \frac{X(0)}{1-rh}$
$X(2h) = \frac{X(h)}{1-rh} = \frac{X(h)}{(1-rh)^2}$
$X(nh) = \frac{X(0)}{(1-rh)^n}$
$X(t) = \frac{X(0)}{(1-rh)^{\frac{t}{h}}}$

1 Mark for sticking in nh for t.
1 Mark for getting the recursion to the 2nd last line
1 Mark for then going back to t
d)

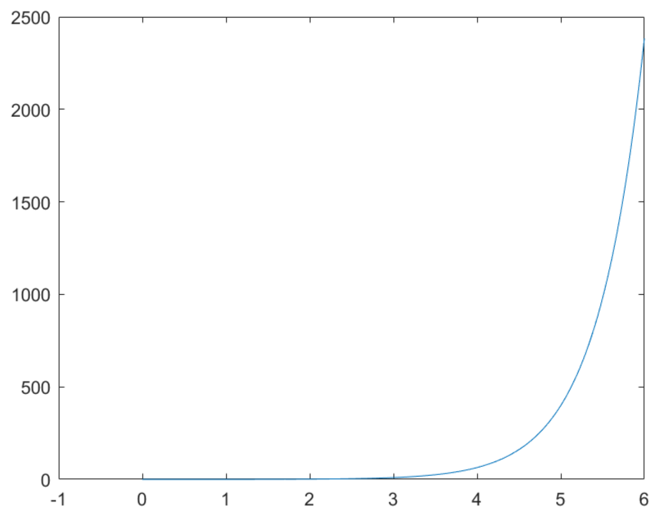


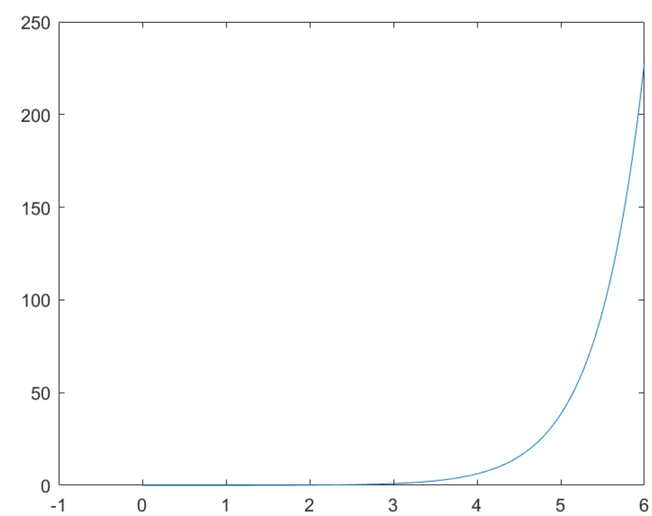


The discretization gets closer as h gets smaller
x(6)=2.9530e+04
For larger h X(6)=3.1913e+04
For smaller h X(6)=2.9757e+04

2 mark for the graph (Note I did them separate, them being on the same graph as asked is correct, if they did separate them like I did here, also correct) (should look different, 1st should be going up higher

since less accurate
1 mark for stating something along the lines of as h gets smaller the discretization is more accurate (or the 2nd graph is closer)
1 mark for the correct output of X(6)'s.
e)

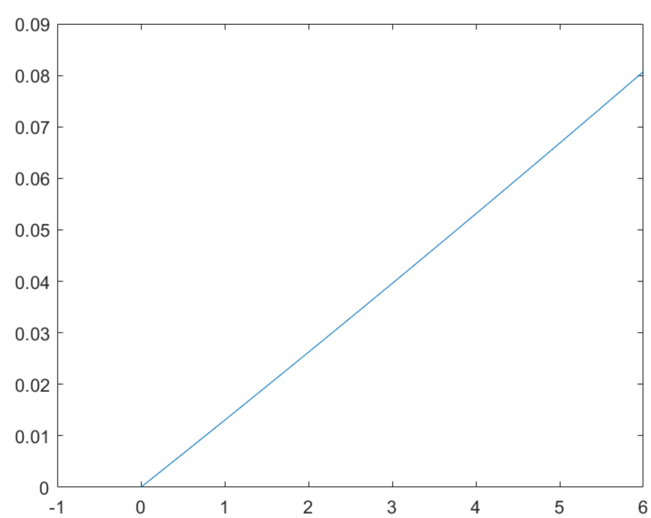


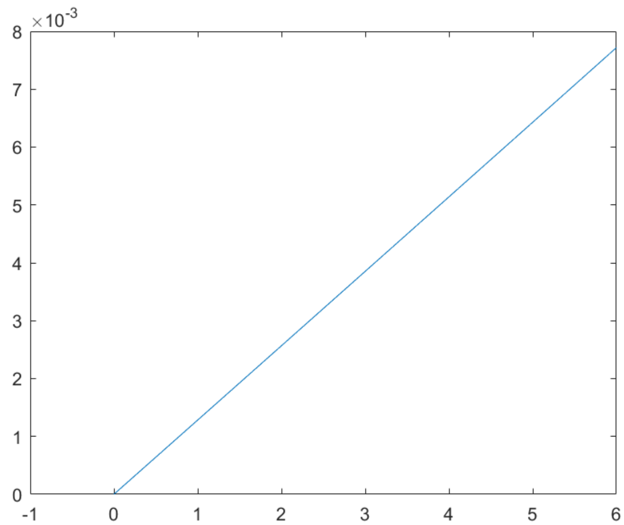


E(t) represents the error in the discretization.
For larger h E(6)=2.3837e+03
For smaller h E(6)=227.9042

As h gets smaller, the discretization becomes more accurate. The error comes from (discretization, machine rounding, machine error, etc).

2 Mark for the graph(s)
1 Mark for E(t) represents the error in the discretization.
1 Mark for the E(6)'s
1 Mark for something again on the lines that as h gets smaller the discretization becomes more accurate.
1 Mark for saying where the error comes from.
f)





As h gets smaller, the discretization becomes more accurate. 2 Marks for the graph(s)
1 Mark for something again on the lines that as h gets smaller the discretization becomes more accurate.

QUESTION 2

Consider the following matrix equation:
$\frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{-1}{2} & 4 \\ -4 & \frac{-1}{2} \end{bmatrix} \vec{x}$
Where $\vec{x} = (x_1(t), x_2(t))$. Let A be the matrix above.
a) Classify this model
b) Find the fixed points of this model
c) Determine the stability of each fixed point. Furthermore describe what you expect the solution to look like near the fixed point.
d) Find the explicit solution $\vec{x(t)}$
Assume that $\vec{x(0)} = (-3, 3)$. e) Using a computer, write down the explicit solution for this initial condition, and graph it as well.

19 Mark Question.
1 for a)
3 for b)
5 for c)
5 for d)
5 for e)

a) This is a Linear, Multivariate, Continuous, Deterministic model
1 mark all 4 parts must be correct

b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & 4 \\ -4 & \frac{-1}{2} \end{bmatrix} \vec{x}$
$0 = \frac{-x_1}{2} + 4x_2$
$0 = -4x_1 + \frac{x_2}{2}$
Solve this or simply use the fact that the matrix has nonzero determinant to get that the fixed point is (0,0).

3 Marks for getting to (0,0)
-1 Mark if they state (0,0) with no reason.
c) Method 1)
Trace=-1
$Det = \frac{65}{4} > 4 * 1 = 4$
Therefore near the fixed point it is expected to be a stable spiral.

Method 2)
Find the eigenvalues of the matrix
$-\frac{1}{2} \pm 4i$
Therefore it'll be a stable spiral.

I assume most will use Method 2.
3 Marks for getting to the correct eigenvalues.
1 Mark for saying it's stable since real part is less than 0
1 Mark for further stating that it is a spiral
d) $A - (-\frac{1}{2} - 4i) = \begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix}$
$4ix_1 + 4x_2 = 0$
$-4x_1 + 4ix_2 = 0$
Multiply first equation by i to get $-4x_1 + 4ix_2 = 0$, which is the second equation. Therefore $x_1 = ix_2$, therefore eigenvalue 1 is (1,i) and eigenvalue 2 is (1,-i). Important to note if they have ANY linear multiplier of these it would still be correct.
Therefore $\vec{a} = (1, 0)$ $\vec{b} = (0, 1)$ Note they could have b be (0,-1) since there is no order to the eigenvalues.
Therefore the general solution is:
$x(\vec{t}) = e^{\lambda t}(c_1(cos(\mu t)\vec{a} - sin(\mu t)\vec{b}) + c_2(sin(\mu t)\vec{a} + cos(\mu t)\vec{b}))$
$x(\vec{t}) = e^{\frac{-1}{2}t}(c_1(cos(4t)(1, 0) - sin(4t)(0, 1)) + c_2(sin(4t)(1, 0) + cos(4t)(0, 1)))$
They do not need to solve for c_1, c_2 as it was mentioned in class they will NEVER have to do that without a computer.

4 Marks for getting to the eigenvectors.
1 Mark for then sticking in the eigenvectors into the general form.
If they attempted to solve for c_1, c_2 ignore that part.
e)$(-3,3) = \vec{x(0)} = e^{\frac{-1}{2}t}(c_1(cos(4t)(1,0) - sin(4t)(0,1)) + c_2(sin(4t)(1,0) + cos(4t)(0,1)))$
$= c_1(1,0) + c_2(0,1)$
Therefore $c_1 = -3, c_2 = 3$
Therefore $\vec{x(t)} = e^{\frac{-1}{2}t}(-3(cos(4t)(1,0) - sin(4t)(0,1)) + 3(sin(4t)(1,0) + cos(4t)(0,1)))$

```
% function [Ydot]=newrhs(~,Y)
% Ydot(1)=-1/2*Y(1)+4*Y(2);
% Ydot(2)=-4*Y(1)-1/2*Y(2);
% Ydot=Ydot';

2nd file:
opts = odeset('RelTol',1e-8,'AbsTol',1e-12);
sol=ode45(@newrhs,[0 200],[-3 3],opts);
plot(sol.x,sol.y)
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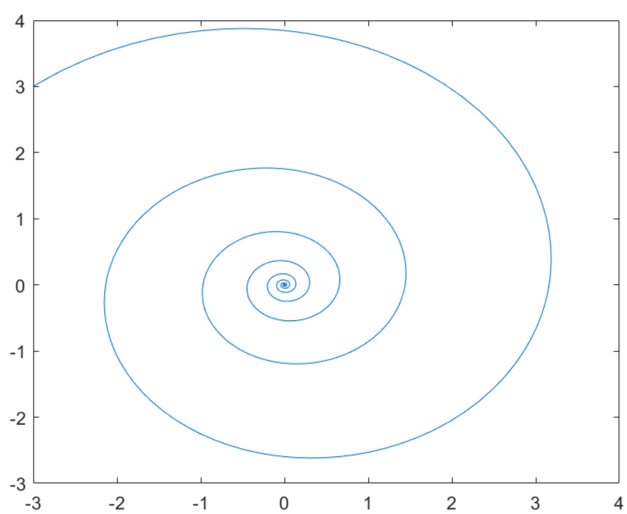

2 marks for solving for c_1, c_2
1 marks for the graph being in x1 vs x2 form
2 marks for the spiral like shape of the graph shown above