The basic model is now
$\frac{d\vec{x}}{dt} = A\vec{x}$
where $\vec{x} = \vec{x}(t)$ is an n dimensional vector and A is a n by n matrix. As before we find fixed points by setting $\frac{d\vec{x}}{dt}$=0 and solving. However stability now gets more complicated. One important part of this is we now allow complex eigenvalues.

1. The Matrix Exponential

The solution to our basic model in this section is often written as $\vec{x}(t) = \vec{x}(0)e^{At}$, which is known as a matrix exponential. The only way we know how to calculate a matrix exponential is by series expansion of the exponential function:
$e^{Ax} = 1 + Ax + \frac{(Ax)^2}{2} + \frac{(Ax)^3}{3!} + \cdots$
so when the exponent is now a matrix instead we get:
$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{3!} + \cdots$
where I is the n by n identity matrix. Furthermore if we assume that A can be diagonalized ($A = SDS^{-1}$), we get the following:
$e^{SDS^{-1}t} = I + SDS^{-1}t + \frac{(SDS^{-1}t)^2}{2} + \frac{(SDS^{-1}t)^3}{3!} + \cdots$
with some simplification then becomes:
$e^{SDS^{-1}t} = S(I + Dt + \frac{(Dt)^2}{2} + \frac{(Dt)^3}{3!} + \cdots)S^{-1}$
Just like before, taking the power of a diagonalized matrix is the the power of the elements of that matrix, so everything inside of the brackets becomes easy to compute.

2. eigenvalue approach

In order to solve for $\vec{x}$, we end up mimicking what we did with multivariate discrete models. First we diagonalize A, but writing it as SDS^{-1}, where the columns of S are the eigenvectors $\vec{v_1}, \vec{v_2}, \cdots \vec{v_n}$ of A, and D is simply a diagonal matrix, where the elementals on the diagonal are the eigenvalues $d_1, d_2, \cdots d_n$ of A. We now get a new form for the explicit solution which looks as follows:
$\vec{x}(t) = c_1e^{d_1t}\vec{v_1} + c_2e^{d_2t}\vec{v_2} + \cdots + c_ne^{d_nt}\vec{v_n}$
Where the $c_i's$ are the constant coefficients which are then solved using

the given initial conditions.

3. A complex example

We start by looking at a simple two dimensional example.

A(t)=$\begin{bmatrix} 0 & 5 \\ -5 & 6 \end{bmatrix}$

This can be expanded out into the following 2 dimensional system:

$\frac{dx_1}{dt} = 5x_2$

$\frac{dx_2}{dt} = -5x_1 + 6x_2$

To figure out the fixed point of the system, we simply note that $det(A) = -15$ since this value isn't 0, A is invertible, and the only fixed point is the point (0,0). To find the eigenvalues of A we look at $\lambda I - A$ (or you could look at $A - \lambda I$) which is:

$(\lambda I - A(t)) = \begin{bmatrix} \lambda & -5 \\ 5 & \lambda - 6 \end{bmatrix}$

To find the eigenvalue of this we now look at $det(\lambda I - A) = 0$

$(\lambda)(\lambda - 6) + 25 = 0$

$\lambda^2 - 6\lambda + 25 = 0$

$\lambda_i = \frac{6 \pm \sqrt{36-100}}{2}$

$\lambda_i = 3 \pm 4i$

Where i is the imaginary number satisfying $i^2 = -1$. Note if the entries of the original matrix A are real, and if you get a complex eigenvalue, you are guaranteed to get it's complex conjugate, and the exact same thing happens for the eigenvector. i.e if $d_1 = c + di$ is an eigenvalue of A, then $d_2 = c - di$ is an eigenvalue of A as well. If $v_1 = a + bi$ is an eigenvalue of A, then $v_2 = a - bi$ is also an eigenvalue of A. Note this will also makes the c_i's complex conjugates as well

Now going back to our explicit solution the first term has $e^{d_1 t}$ which becomes $e^{(3+4i)t}$. Now using Euler's identity we now get:

$e^{(3+4i)t} = e^{3t}(cos(4t) + isin(4t)) = e^{3t}cis(4t)$

Similarly:

$e^{(3-4i)t} = e^{3t}(cos(4t) - isin(4t))$

This is due to the fact that $cos(-x) = cos(x)$ and $sin(-x) = -sin(x)$

Therefore we have:

$x(t) = c_1(e^{3t}cis(4t))\vec{v_1} + c_2(e^{3t}(cos(4t) - isin(4t)))\vec{v_2}$

$=e^{3t}(cos(4t)(c_1\vec{v_1} + c_2\vec{v_2}) + isin(4t)(c_1\vec{v_1} - c_2\vec{v_2}))$

Let $\vec{a} = Re(\vec{v_1}) = Re(\vec{v_2})$ and $\vec{b} = Im(\vec{v_1}) = -Im(\vec{v_2})$

and let $c_3 = Re(c_1) = Re(c_2)$ and $c_4 = Im(c_1) = -Im(c_2)$

Then we have:

x(t)=$e^{3t}(cos(4t)((c_3+ic_4)(\vec{a}+i\vec{b})+(c_3-ic_4)(\vec{a}-i\vec{b}))+isin(4t)((c_3+ic_4)(\vec{a}+i\vec{b})+(-c_3+ic_4)(\vec{a}-i\vec{b}))$
$=e^{3t}(cos(4t)(2c_3\vec{a}-2c_4\vec{b})+sin(4t)(-2c_3\vec{b}-2c_4\vec{a}))$
Now if we let $c_5=2c_3$ and $c_6=-2c_4$ then we get:
x(t)=$e^{3t}(c_5(cos(4t)\vec{a}-sin(4t)\vec{b})+c_6(sin(4t)\vec{a}+cos(4t)\vec{b}))$
Where c_5 and c_6 can be solved from the initial conditions

4. General case

If we now let our eigenvalue be $\lambda+i\mu$ and it's eigenvector $a+bi$ then the solution is:
x(t)=$e^{\lambda t}(c_1(cos(\mu t)\vec{a}-sin(\mu t)\vec{b})+c_2(sin(\mu t)\vec{a}+cos(\mu t)\vec{b}))$
where we have simply reset the coefficients to a lower number. To summarize, the explicit solution is found by first finding the eigenvalues d_1 and d_2. Then their real and imaginary parts λ and μ respectively. Next, find the eigenvectors $\vec{v_1}$ and $\vec{v_2}$, and their real and imaginary parts $\vec{a}$ and $\vec{b}$ respectively. Then compose the solution from the expression above. If an initial condition is given, use it to determine the arbitrary constants c_1 and c_2.

5. Stability

Now that we actually have an explicit solution, the stability is straight forward. Since this is an m dimensional system, the system itself will be stable iff all eigenvalues have a negative real part. That means that $\lambda<0$ for stability. This has nothing to do with the dominate eigenvalue like before, and needs every eigenvector to have a negative real part. If a single eigenvalue has a positive real part then the fixed point is unstable.