Question 1

Consider the following population model:
$\frac{dx}{dt} = rx(1 - e^{bx})$
a) Classify the model
b) Find the fixed point(s)
c) Find the stability of these fixed point(s)
d) Assuming that $x(0) = x_0$, what happens to the population over time?

Answer:
a) Nonlinear Univariate Continuous Deterministic
b)$0 = rx(1 - e^{bx})$. Therefore $x^* = 0$
c) $J = r - re^{bx} - brxe^{bx}$
$J(0) = r - r = 0$ therefor linearization fails.
Looking at a phase diagram. If $x > 0$ then $1 - e^{bx} < 0$ therefore 0 is stable if $r > 0$, unstable if $r < 0$. Since this is a population model you don't have to look at 0 from the left.
d) Population declines to 0.

Question 2

$$T = \begin{bmatrix} .9 & .15 & .25 \\ .075 & .8 & .25 \\ .025 & .05 & .5 \end{bmatrix}$$

a) Classify the model
b) Find the fixed points of the model
c) Find the steady-state matrix L.

Answer:
a) Linear Multivariate Discrete Stochastic (Discrete since it's a Markov Chain)
b) We know that we have one eigenvalue that is 1, so check for it's corresponding eigenvector.

$$T - I_3 = \begin{bmatrix} -.1 & .15 & .25 \\ .075 & -.2 & .25 \\ .025 & .05 & -.5 \end{bmatrix}$$

row reduce this matrix to get: $\begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

Therefor our eigenvector is (10,5,1) and we want the entries to be a probability so they have to add up to 1.
Therefore the eigenvector we want is $(\frac{10}{16}, \frac{5}{16}, \frac{1}{16})$.
c) Using b) the steady-state matrix L is:

$$T = \begin{bmatrix} \frac{10}{16} & \frac{10}{16} & \frac{10}{16} \\ \frac{5}{16} & \frac{5}{16} & \frac{5}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \end{bmatrix}$$

Question 3

Consider the following population model:
$\frac{dx}{dt} = rx(1 - \frac{y}{k} - \frac{ax}{k})$
$\frac{dy}{dt} = ry(1 - \frac{x}{k})$
a) Classify the model
b) Find the fixed points
c) Determine the stability of each fixed point
d) If r=1, a=.5, k=1 describe the solution near each fixed point

Answer:
a) Nonlinear Multivariate Continuous Deterministic
b) $0 = rx(1 - \frac{y}{k} - \frac{ax}{k})$
$0 = ry(1 - \frac{x}{k})$
The fixed points are $(0, 0), (\frac{k}{a}, 0), (k, k(1 - a))$
c) Note you could remove the r since it does nothing for stability.

$$J = \begin{bmatrix} r - \frac{ry}{k} - \frac{2rax}{k} & \frac{-rx}{k} \\ \frac{-ry}{k} & r - \frac{rx}{k} \end{bmatrix}$$

$$J(0, 0) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$J(\frac{k}{a}, 0) = \begin{bmatrix} -r & \frac{-r}{a} \\ 0 & \frac{ar-r}{a} \end{bmatrix}$$

$$J(k, k(1 - a)) = \begin{bmatrix} -ra & -r \\ r(a - 1) & 0 \end{bmatrix}$$

(0,0) is stable if $r < 0$
$(\frac{k}{a}, 0)$ is stable if $r > 0$ and $0 < a < 1$
$(k, k(1 - a))$ is stable if the trace is negative and the determinant is positive, so $a > 0, r > 0, a > 1$, therefore $a > 1$ and $r > 0$.
d) (0,0) Trace=2 Det=1, therefore $Trace^2 - 4Det = 0$, therefore nearby the fixed point is a degenerate unstable node. (unstable node would

earn full marks as well).
(2,0) Trace=-2 Det=-1(-1)=1 therefore $Trace^2 - 4Det = 0$, therefore nearby the fixed point is a degenerate stable node. (stable node would earn full marks as well).
(1,.5) Trace=-.5 Det=-.5 therefore since $Det < 0$, nearby the fixed point is a saddle.

Question 4

Consider the following System
$N(t+1) = (1 + \frac{1}{r})N(t)$
Where r is a Bernoulli random variable taking one of the two values r_1, r_2 with probability
$P[r = r1] = p$ and $P[r = r2] = 1 - p$
a) Classify the model
b) Find the fixed point(s)
c) Find the stability of the fixed point(s)
Now assume that $r1 = -5, r2 = 2$ and $p = 0.1$.
d) Find the fixed point(s)
e) Find the stability of the fixed point(s)

Answer:
a) Linear Univariate Discrete Stochastic
b) Let $N(t+1) = N(t) = N^*$ then $0 = \frac{1}{r}N^*$ then since r is a stochastic variable and isn't infinity, $N^* = 0$ is the fixed point.
c) $E(N(t)) = ((1 + \frac{1}{r1})^p(1 + \frac{1}{r2})^{1-p})^t N(0)$
Therefore 0 will be stable if $((1 + \frac{1}{r1})^p * (1 + \frac{1}{r2})^{1-p}) < 1$
d) Still 0
e) $(1 + \frac{1}{-5})^{.1}(1 + \frac{1}{2})^{.}9 = .8^{.1}1.5^{.9} > 1$ (You can look at this to the 10th power, and note it as $.81.5^9$ which is way bigger than 1).
Therefore the fixed point is unstable.

Question 5

Consider the system:
$\frac{dx}{dt} = -(x - y)(1 - x - y)$
$\frac{dy}{dt} = x(2 + y)$
a) Classify the model
b) Find the fixed points
c) Determine the stability of each fixed point
d) Describe the solution near each fixed point

Answer:
a) Nonlinear Multivariate Continuous Deterministic
b) $0 = -(x-y)(1-x-y)$
$0 = x(2+y)$
Fixed points are $(0,0),(0,1),(-2,-2),(3,-2)$
c) $J = \begin{bmatrix} 2x-1 & 1-2y \\ 2+y & x \end{bmatrix}$
$J(0,0) = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$
$J(0,1) = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$
$J(-2,-2) = \begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}$
$J(3,-2) = \begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix}$
(0,0) has $Trace < 0$ and $Det < 0$ therefore unstable
(0,1) has $Trace < 0$ and $Det > 0$ therefore stable
(-2,-2) has $Trace < 0$ and $Det > 0$ therefore stable
(3,-2) has $Trace > 0$ and $Det > 0$ therefore unstable
d) (0,0) has $Det < 0$ therefore unstable saddle near the fixed point
(0,1) has $Trace = -1$ and $Det = 3$ so $Trace^2 - 4Det = -11 < 0$ therefore stable spiral near the fixed point
(-2,-2) has $Trace = -7$ and $Det = 10$ so $Trace^2 - 4Det = 9 > 0$ therefore stable node near the fixed point
(3,-2) has $Trace = 8$ and $Det = 15$ so $Trace^2 - 4Det = 4 > 0$ therefore unstable node near the fixed point

Question 6

Consider the system:
$N(t+1) = \frac{aN(t)}{1+bN(t)}$
a) Classify the model
b) Find the fixed points
c) Determine the stability of each fixed point

Answer:
a) Nonlinear Univariate Discrete Deterministic
b) Let $N(t) = N(t+1) = N^*$ then $N^* = 0$ or $1 = \frac{a}{1+bN^*}$. Therefore $N^* = 0$ or $\frac{a-1}{b}$
c) $f(N) = \frac{aN}{1+bN}$

$f'(N) = a(bN+1)^2$
$f'(0) = a$
$f'(\frac{a-1}{b}) = \frac{1}{a}$
0 is stable if $|a| < 1$
$\frac{a-1}{b}$ is stable if $|a| > 1$