Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$. How large is the error if we approximate the sum s by the partial sum s_{100}?

$\frac{1}{n^2}$ is positive, continuous, and decreasing for $n \geq 1$.

$$\int_n^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_n^{\infty} = \frac{1}{n}$$

$$\frac{1}{101} \leq s - s_{100} \leq \frac{1}{100}$$

$\therefore$ The error is approximately $1/100$.

If we want the error to be less than $.005$, how many terms do we add up?

Comparison Test

Let $\{a_n\}$ be a sequence with non negative terms.

$$S_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n-1} a_i + a_n = S_{n-1} + a_n$$

$\therefore$ $S_1 \le S_2 \le S_3 \le \ldots$ (monotonic increasing)

so $\{s_n\}$ converges, and thus $\sum a_i$ converges,

iff $\{S_n\}$ is bounded.

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Let $\sum a_n$ and $\sum b_n$ be two series such that

$0 \le a_n \le b_n$ for all n.

Then $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

$\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

Examples

1) a) $\sum_{n=1}^{\infty} \frac{3}{n^2+2n+1}$ b) $\sum_{n=1}^{\infty} n^2/(n^3+1)$

c) $\sum_{n=1}^{\infty} \ln(n)/n$ d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+n+1}}$

Limit Comparison Test

If $\sum a_n$, $\sum b_n$ are series with positive terms,

and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is finite,

the either both series converge, or both series diverge.

Example $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$